

NOTES
ON
THEORETICAL AERODYNAMICS
AND
AIRFOIL THEORY

A THESIS
SUBMITTED FOR THE DEGREE
OF
MASTER OF SCIENCE IN AERONAUTICAL ENGINEERING
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INTRODUCTION.

In the last twenty years the field of aerodynamics has been made into a science by such outstanding men as Joukowski, Prandtl, von Karman, Glauert, Theodorsen and others. One of the important contributions of these men has been the theory for computing the lift of airfoils whose profile contours can be represented by a mathematical expression or are entirely arbitrary in shape. The mathematical basis of this theory is the "Conformal Transformation". A mastery of the elementary theory of complex numbers is a prerequisite to the understanding of a conformal transformation and because this theory is not commonly taught in an undergraduate course the most capable undergraduates have great difficulty in obtaining even a slight conception of the theory of lift of an airfoil. In fact they find the theory presented in such a concise manner that they do not have the courage to attempt to understand it. It seems to the writer that this unfortunate situation should not be permitted to continue. The present thesis is written with the hope that, because of its mathematical detail and step by step approach to the desired end, at least our interested students can obtain a sufficient grasp of the theory to encourage them to seek the profound sources for the complete theory.

In addition to the theory of complex numbers, a student needs a knowledge of some of the elements of classical hydromechanics before he can attempt a study of the theory of lift. Consequently,

an attempt has been made herein to present in as brief a manner as possible the necessary facts about: (1) stream functions and velocity potentials of rectilinear and vortex motions, sources and sinks, doublets, and simple combinations thereof; (2) Bernoulli's theorem; (3) circulation; and (4) the several criteria for perfect fluid flow. The writer does not have such a wide familiarity with the literature on these subjects that he can say with certainty what in this thesis is original and what is not. He believes, however, that originality may be found in the treatment of Bernoulli's theorem, the stream function of a doublet, the explanation of a conformal transformation, the lift of a flat plate, and minor details.

The writer wishes to express appreciation to Prof. Montgomery Knight for his encouraging influence in preparation of this thesis, for his careful reading of the manuscript, and for his many constructive criticisms. The writer also feels so much gratitude toward Dr. D.M. Smith for an undergraduate and graduate training in mathematics which made possible an understanding of the sources of this thesis that acknowledgment must be made herein.

The sources of this thesis are Reid's "Applied Wing Theory", Glauert's "Airfoil and Airscrew Theory", Glauert's "A Theory of Thin Airfoils" (R & M 910 of the Aeronautical Research Committee for 1924-25), Theodorsen's "The Theory of Wing Sections of Arbitrary Shape" (N. A. C. A. Report # 411, 1931), and notes taken in graduate mathematics under Dr. D.M. Smith.

ASSUMPTIONS

In the discussion which follows air will be considered an incompressible, homogeneous, continuous, and frictionless fluid. It will be considered as occupying so great a space that at a point where its motion is being studied every exterior boundary is too remote to affect the motion. Every motion will be considered a steady motion; i.e., an observer situated at any point will observe that every particle of air passing through that point will have a velocity identical with that of every one which preceded it. Whatever may be the motions of the particles at the different points in a plane the particles at the corresponding points in all parallel planes will have identical motions. Finally, to preserve our conceptions of the physical quantities involved the air will be taken between two of these parallel planes one foot apart. Hence taking the plane of the paper as the plane of motion a vector representing the velocity of a particle of air at a point also represents the velocity of every particle on a line one foot long through that point perpendicular to the paper.

STREAM LINE, STREAM TUBE, ETC.

A line in the plane of motion tangent to the vectors representing the successive velocities of a particle of air as it moves along the plane is defined as a stream line. The stream lines passing through the boundary of a closed curve in any plane perpendicular to the plane of motion form a stream tube. The section area of a stream tube varies with the velocity of the fluid elements flowing in the tube but from the definition of a stream line no fluid element can have a component of Vel. perpendicular to the walls of the stream tube. It follows from

this that the fluid elements contained in one stream tube can not leave that tube at any point and thereafter flow in a new tube. A stream tube may be visualized as a material pipe varying in cross-sectional area and direction in exact conformity with the stream tube. From this then the area of the cross-section of the stream tube will change with the mean velocity of the fluid increasing when the mean velocity decreases and vice-versa. Hence, the velocity of the fluid may be estimated by the distance between adjacent stream lines, being inversely proportional to that distance. All these deductions are based on the formula

$$Q = AV$$

Q = Quantity of air flowing in a stream tube in cu. ft. *per sec.*

A = Area of cross-section of a stream tube in sq. ft.

V = Mean Vel. of fluid in ft. per sec.

STREAM FUNCTION.

Select any point in the plane of motion as an origin and draw a line from the origin to any point P in the same plane. This line represents an area with the length of the line as one dimension and the length of a line one foot long perpendicular to the plane of motion as the other. With fluid flowing with steady motion the quantity of fluid crossing this area in cu. ft. per sec. represents the stream function, ψ , at the point P .

A particular value of the stream function may be either positive or negative according to the arbitrary sign we wish to give it.

In this discussion a positive stream function will be defined in the following manner: if an observer at the origin facing the point P notes that the fluid is coming from his left and passing through

the area toward his right then the stream function at P_1 will be taken positive.

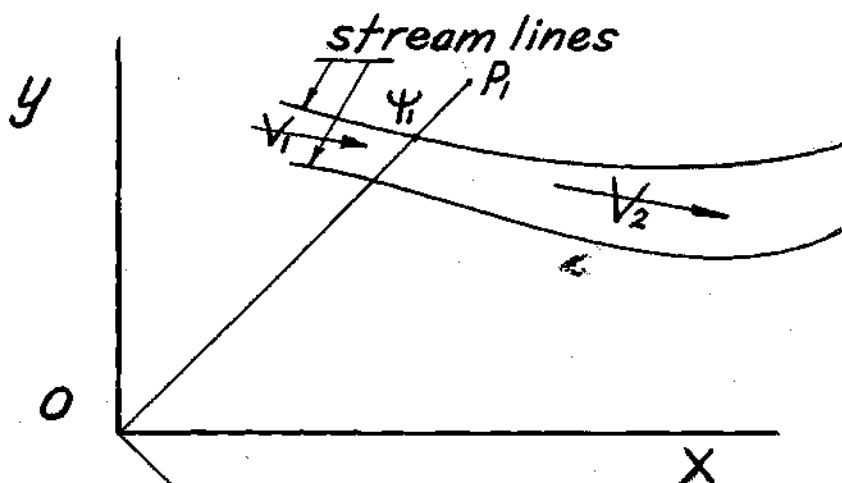
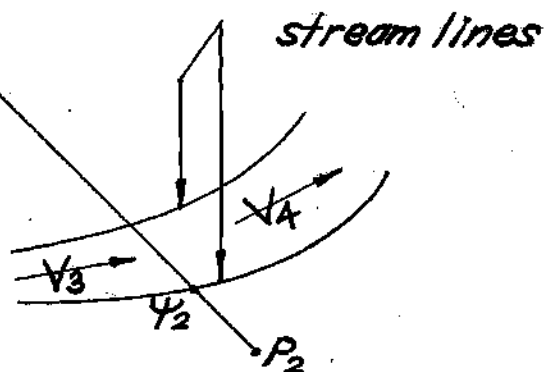


Fig. 1

Ψ_1 is positive.
 Ψ_2 is negative



x and y Components of Vel. at a point.

Let $\begin{cases} u = \text{component of Vel. parallel to } x \text{ axis.} \\ v = \text{component of Vel. parallel to } y \text{ axis.} \end{cases}$

Assume $\Psi = F(x, y)$ to be a continuous function with continuous partial derivatives.

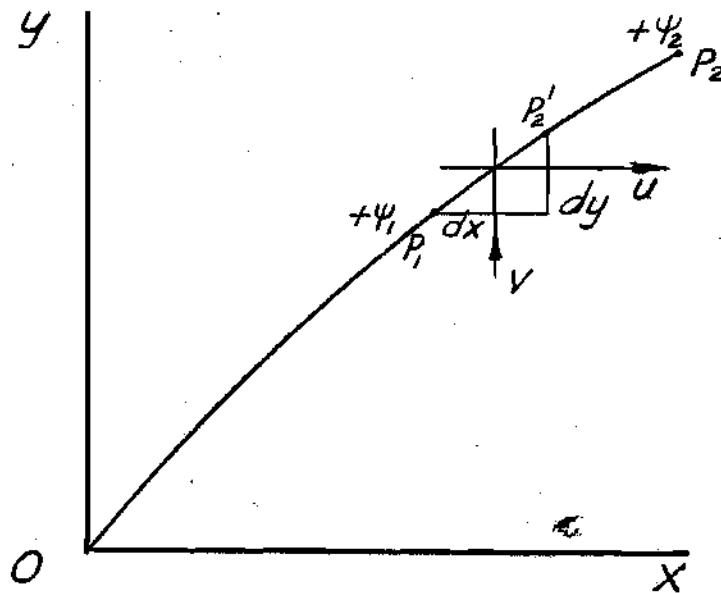


Fig. 2

$$\psi_2 > \psi_1$$

$$\therefore \psi_2 - \psi_1 = +\Delta\psi \text{ (an increment in } \psi \text{)}$$

Let P_2 approach P_1 until it is an infinitesimal distance from P_1 and at P_2' . Then $\Delta\psi$ will become $d\psi$.

$$\begin{cases} dy \text{ represents an area } dy \cdot 1' \\ dx \text{ represents an area } dx \cdot 1' \end{cases}$$

since $Q = AV$

$$\begin{cases} udy = \text{quantity of fluid in cu. ft. per sec. passing} \\ \quad \text{through the area } dy \cdot 1' \\ vdx = \text{quantity of fluid in cu. ft. per sec. passing} \\ \quad \text{through the area } dx \cdot 1' \end{cases}$$

A positive x component of Vel. will cause a positive flow across the area $P, P_2' \times 1$ ft. if dy is positive but a positive y component of Vel. will cause a negative flow across the same area if dx is positive. Hence $(udy - vdx)$ is the flow across the area $P, P_2' \times 1$ ft. which is by hypothesis positive.

$$\therefore d\psi = udy - vdx$$

But from calculus

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

From which

$$udy - vdx = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

But because x and y are independent variables these equations can not be equal unless the coefficients of like variables are equal.

$$\begin{cases} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} \end{cases} \quad [1]$$

Stream Function for a uniform stream parallel to x axis flowing in the direction of positive x .

For this case $\begin{cases} \frac{\partial \psi}{\partial x} = 0 \\ \frac{\partial \psi}{\partial y} = U \end{cases}$

Hence $\int \frac{\partial \psi}{\partial y} dy = \int U dy = Uy + \text{constant}$

This Equation tells us that on every axis we draw parallel to the y axis the stream functions at the points along it will be identical with those of the corresponding points on the y axis.

Likewise $\int \frac{\partial \psi}{\partial x} dx = \int 0 dx = \text{constant}$

Let $\psi = 0$ where $y = 0$

then $\psi = Uy$

y	
16	16
12	12
8	8
4	4
0	x
-4	-4
-8	-8
-12	-12
-16	-16

Fig. 3

Fig 3. represents the stream lines for a flow

$$\psi = 4y$$

Where the distance between the lines to scale represents one foot. i. e. $U = 4 \text{ f.p.s.}$

Stream Function for a uniform stream parallel to y axis flowing in the direction of negative y .

For this case

$$\begin{cases} -\frac{\partial \psi}{\partial x} = -V \\ \frac{\partial \psi}{\partial y} = 0 \end{cases}$$

$$\int \frac{\partial \psi}{\partial y} dy = \int 0 dy = \text{constant.}$$

$$\int \frac{\partial \psi}{\partial x} dx = \int V dx = Vx + \text{constant.}$$

Let $\psi = 0$ where $x = 0$

Then $\psi = Vx$

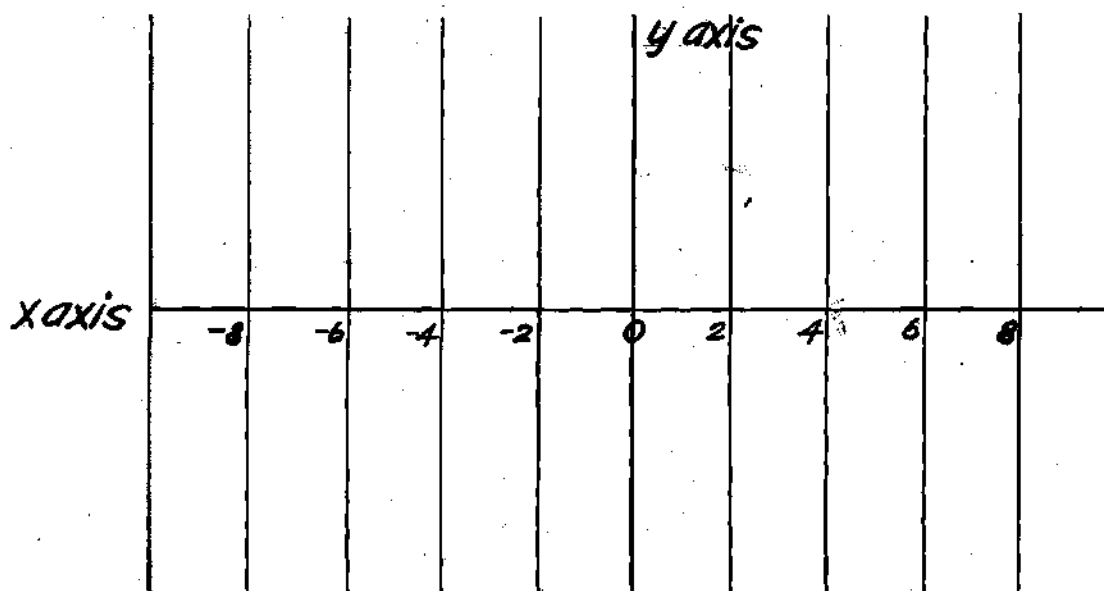


Fig. 4

Fig. 4 represents the stream lines for a flow

$$\psi = 2x$$

Where the distance between the lines represents 1' foot to scale.

$$\text{i. e. } V = 2 \text{ f.p.s.}$$

Stream Function for a uniform stream having a Vel. which has a constant positive component (U) parallel to the x axis and a constant negative component (V) parallel to the y axis.

$$\begin{cases} \frac{\partial \psi}{\partial y} = +U \\ \frac{\partial \psi}{\partial x} = -V \end{cases}$$

$$\int \frac{\partial \psi}{\partial y} dy = \int U dy = Uy + A$$

$$\int \frac{\partial \psi}{\partial x} dx = \int -V dx = -Vx + B$$

Then

$$\psi = Uy - Vx + A + B$$

$$\text{Let } \psi = 0 \text{ where } \begin{cases} x = 0 \\ y = 0 \end{cases}$$

Now for the x axis ($y=0$) the quantity of fluid passing under the axis between the origin and x , i. e. through an area x ft. long and 1 ft. perpendicular to xy plane, must just equal Vx .

Likewise for the y axis ($x=0$) the quantity of fluid flowing under the y axis between 0 and y , must just equal Uy . Both of the preceding statements are based on $Q = AV$

$$\therefore A = B = 0.$$

and $\psi = Uy + Vx$

But this Equation could have been obtained by simply adding the two simple cases previously derived. In general, complex stream functions may be derived by superposing simple stream functions until the desired one is obtained. This is a very important and useful principle.

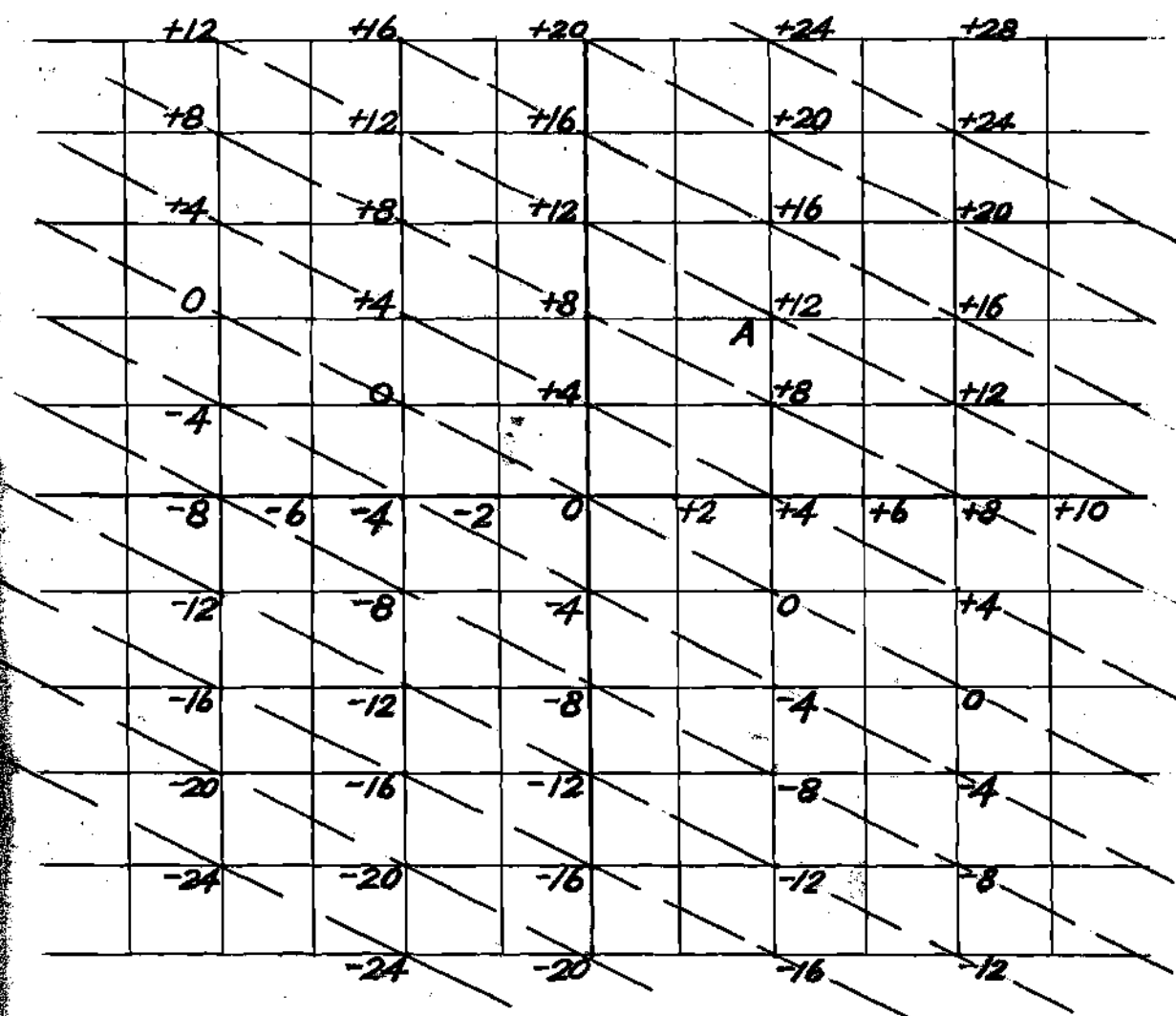


Fig. 5

In Fig. 5

The horizontal stream lines are for $\psi = 4y$

The vertical stream lines are for $\psi = 2x$

The inclined stream lines (shown broken) are for $\psi = 2x + 4y$

To obtain the value of the stream function at point A add the value on the x axis under it to the value on the y axis opposite it.

It is often more convenient to write the Equation for ψ in polar co-ordinates.

Let V' = component of Vel. perpendicular to radius drawn from origin to any point.

and u' = component of Vel. parallel to radius drawn from origin to any point.

V' is taken to be positive if an observer looking along its vector toward the arrow head observes that his left arm is on the side of the radius leading to the origin. If an observer followed a particle moving with a positive V' around a circle with the origin as a center his left arm would always be on the inside of the turn. u' is taken to be positive if the arrow of its vector is pointing outward away from the origin.

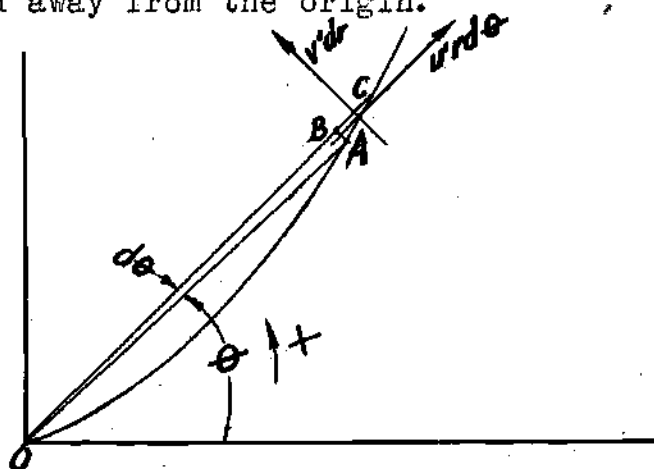


Fig. 6.

Let $\begin{cases} OA = r \\ \theta = \text{a positive angle} \end{cases}$

Then $\begin{cases} BC = +dr \\ \angle AOB = +d\theta \end{cases}$

Also $\begin{cases} u'r d\theta = \text{positive flow under AC or under AB, } (r d\theta) \\ \text{caused by a positive } u'. \\ v'dr = \text{negative flow under AC or under BC, } (dr) \\ \text{caused by a positive } v'. \end{cases}$

Hence $d\psi = u'r d\theta - v'dr$

But $\psi = F(r, \theta)$

hence $d\psi = \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial \theta} d\theta$

\therefore Equating coefficients of like independent variables

$$\begin{cases} \frac{\partial \psi}{\partial \theta} = u'r \\ \frac{\partial \psi}{\partial r} = -v' \end{cases} \quad [2]$$

Source, Sink.

Out of a source fluid is appearing at a uniform rate m (cu. ft. per sec.) for every foot of a line indefinitely long perpendicular to the plane of motion. The fluid will spread out parallel to the plane of motion in all directions with a velocity u' , at every point of a circle of radius r . This is deduced in the following manner.

$$Q = AV$$

$$m = u'(2\pi r) \times 1'$$

$$u' = \text{velocity of efflux}$$

$$2\pi r \times 1' = \text{area through which fluid is passing at the time rate of } m \text{ cu. ft. per sec.}$$

$$\text{so } u'r = \frac{m}{2\pi}$$

$$v' = 0$$

$$\therefore \int \frac{\partial \psi}{\partial \theta} d\theta = \int \frac{m}{2\pi} d\theta = \frac{m}{2\pi} \theta + \text{constant}$$

$$\int \frac{\partial \psi}{\partial r} dr = \int c \cdot dr = \text{constant}$$

Let $\psi=0$ where $\theta=0$

$$\text{Then } \psi = \frac{m}{2\pi} \cdot \theta$$

This Equation is the stream function of a source. For the case where the fluid disappears at the rate $-m$ we have a negative source or a sink.

Hence the Equation of the stream function for a sink or for a source of strength $-m$ is

$$\psi = -\frac{m}{2\pi} \theta$$

Stream Function for the combination of a source and a sink of equal strength.

Let A be the point where axis of source passes through plane of motion.

Let B be the point where axis of sink passes through plane of motion.

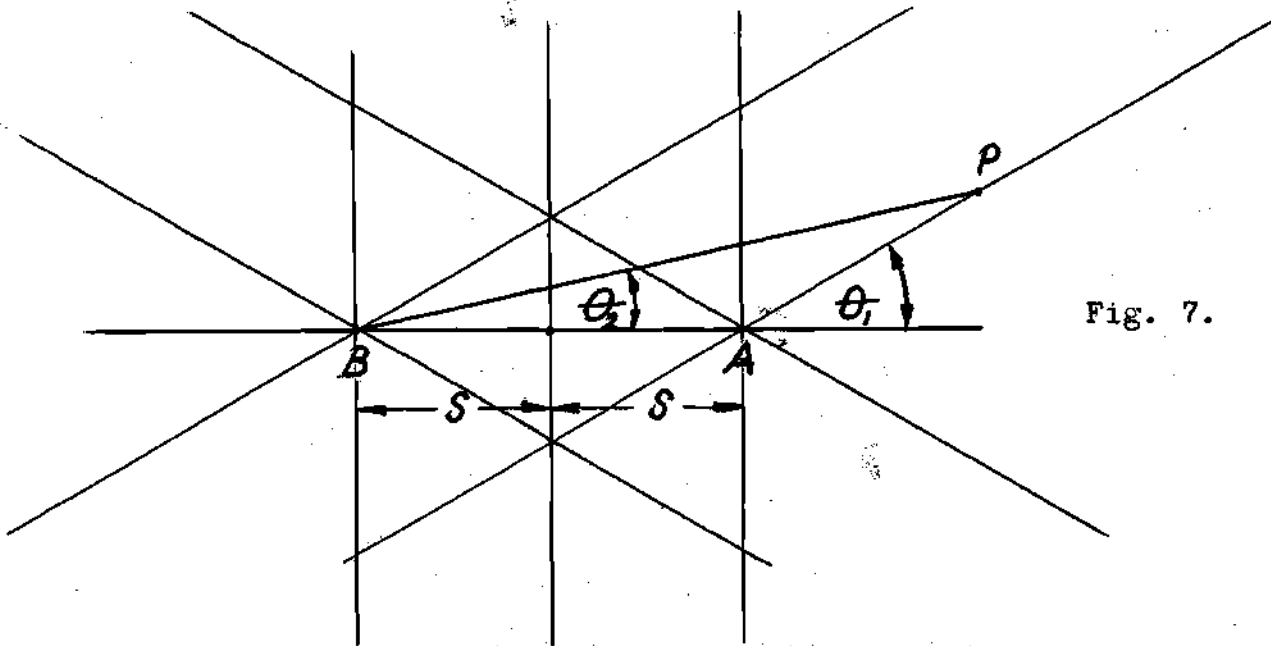


Fig. 7.

Quantity of fluid per sec. passing outward from source between X axis and radial line AP = $\frac{m}{2\pi} \theta_1$

Quantity of fluid passing inward to sink between X axis and radial line BP = $-\frac{m}{2\pi} \theta_2$

$$\therefore \psi = \frac{m}{2\pi} (\theta_1 - \theta_2)$$

$$\tan \theta_1 = \frac{y}{x-s}$$

$$\tan \theta_2 = \frac{y}{x+s}$$

$$\begin{aligned} \text{But } \tan \frac{2\pi\psi}{m} &= \tan(\theta_1 - \theta_2) = \frac{\sin(\theta_1 - \theta_2)}{\cos(\theta_1 - \theta_2)} = \frac{\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2} \\ &= \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} = \frac{\frac{y}{x-s} - \frac{y}{x+s}}{1 + \frac{y}{x-s} \cdot \frac{y}{x+s}} \\ &= \frac{\frac{xy+ys - xy+ys}{x^2-s^2+y^2}}{\frac{x^2-y^2-s^2}{x^2+y^2-s^2}} = \frac{2ys}{x^2+y^2-s^2} \end{aligned}$$

To determine the Equation of a stream line.

Let ψ_1 be the value of ψ at every point on a particular stream line.

$$\tan \frac{2\pi\psi_1}{m} = \text{constant} = A$$

$$A = \frac{2ys}{x^2+y^2-s^2}$$

$$(x^2+y^2-s^2)A - 2ys = 0$$

$$x^2+y^2 - \frac{2s}{A}y = s^2$$

$$x^2+y^2 - \frac{2s}{A}y + \frac{s^2}{A^2} = s^2 + \frac{s^2}{A^2}$$

$$\text{or } x^2 + (y - \frac{s}{A})^2 = (\frac{s}{A} \sqrt{1+A^2})^2 \quad [\text{Equation of stream line}]$$

But this is the Equation of a circle with center at $(0, \frac{s}{A})$.

Where $y = 0$

$$x^2 + \frac{s^2}{A^2} = \frac{s^2}{A^2} (1+A^2) = \frac{s^2}{A^2} + s^2$$

$$\therefore x = \pm s$$

By a very simple analysis then it has been shown that all

the streamlines are circles and all pass through the axis of source and sink.

$$\text{Let } \frac{m}{2\pi} = \frac{q}{\pi}$$

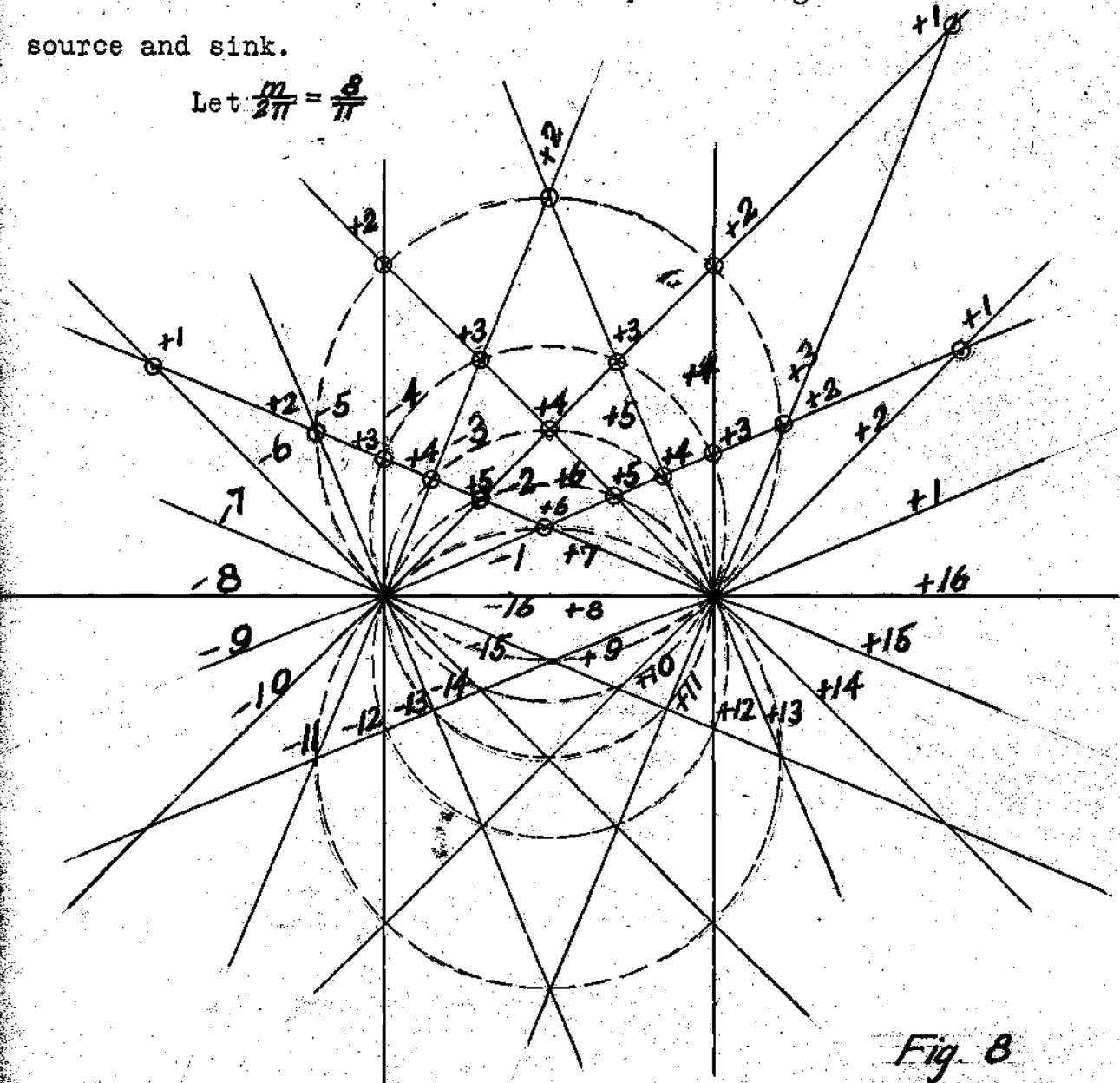


Fig. 8

Graphical construction of streamlines for source and sink of equal strength.

DOUBLET.

Where a source and a sink of equal strength have been brought together until the distance between them is a line of infinitesimal length the resulting stream function is said to be that of a doublet. This stream function will be the limiting one

obtained by allowing s to approach zero while keeping the product of m and s constant in the expression for the stream function of a source and sink of equal strength.

Rewrite $\tan^{-1} \frac{2\pi\psi}{m} = \frac{2ys}{x^2+y^2-s^2}$

as

$$\psi = \frac{m}{2\pi} \tan^{-1} \frac{2ys}{x^2+y^2-s^2}$$

Let $\mu = 2ms = \text{constant} = \text{strength of combination.}$

$$\psi = \frac{\mu}{4\pi s} \tan^{-1} \frac{2ys}{x^2+y^2-s^2}$$

$$\frac{\partial \psi}{\partial y} = u = \frac{\mu}{4\pi s} \cdot \frac{(x^2+y^2-s^2)2s - 2ys \cdot 2y}{(x^2+y^2-s^2)^2 + 4y^2s^2}$$

$$= \frac{\mu}{2\pi} \cdot \frac{x^2-y^2-s^2}{(x^2+y^2-s^2)^2 + 4y^2s^2}$$

$$\frac{\partial \psi}{\partial x} = -v = \frac{\mu}{4\pi s} \cdot \frac{(x^2+y^2-s^2) \cdot 0 - 2ys \cdot 2x}{(x^2+y^2-s^2)^2 + 4y^2s^2}$$

$$v = \frac{\mu}{2\pi} \cdot \frac{2xy}{(x^2+y^2-s^2)^2 + 4y^2s^2}$$

$$q^2 = u^2 + v^2 = \left(\frac{\mu}{2\pi}\right)^2 \frac{(x^2-y^2-s^2)^2 + 4x^2y^2}{[(x^2+y^2-s^2)^2 + 4y^2s^2]^2}$$

$$q = \frac{\mu}{2\pi} \frac{[(x^2+y^2)^2 - 2(x^2-y^2)s^2 + s^4]^{\frac{1}{2}}}{(x^2+y^2-s^2)^2 + 4y^2s^2}$$

A study of the expression for the resultant velocity for the condition that the product, $2ms(\mu)$, is to remain constant shows that for points remote from source and sink the resultant velocity is affected but little by large changes in S .

Rewrite

$$\psi = \frac{\mu}{4\pi S} \tan^{-1} \frac{2yS}{x^2 + y^2 - S^2}$$

as

$$\psi S = \frac{\mu}{4\pi} \tan^{-1} \frac{2yS}{x^2 + y^2 - S^2}$$

taking the partial derivative of both sides with respect to S .

$$\begin{aligned} \psi + S \frac{\partial \psi}{\partial S} &= \frac{\mu}{4\pi} \cdot \frac{(x^2 + y^2 - S^2) \cdot 2y + 2yS \cdot 2S}{(x^2 + y^2 - S^2)^2} \\ &= \frac{\mu}{4\pi} \frac{2y(x^2 + y^2) + 2yS^2}{(x^2 + y^2 - S^2)^2 + 4y^2S^2} \end{aligned}$$

Let $S \rightarrow 0$ and if $\frac{\partial \psi}{\partial S}$ remains finite the stream function approaches

$$\psi = \frac{\mu}{2\pi} \frac{y}{(x^2 + y^2)}$$

Taking the partial derivative with respect to S of

$$\psi = \frac{\mu}{4\pi S} \tan^{-1} \frac{2yS}{x^2 + y^2 - S^2}$$

and then applying L'Hospital's method for evaluation of indeterminate quantities it is found that

$$-\frac{\partial \psi}{\partial S} \rightarrow 0 \text{ as } S \rightarrow 0$$

This being true the above Equation represents the stream function of a doublet.

Futhermore for the doublet

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= u = \frac{\mu}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} \\ \frac{\partial \psi}{\partial x} &= -v = \frac{\mu}{2\pi} \frac{-2xy}{(x^2 + y^2)^2} \end{aligned}$$

and since

$$q^2 = u^2 + v^2 = \left(\frac{\mu}{2\pi}\right)^2 \frac{x^4 - 2x^2y^2 + y^4 + 4x^2y^2}{(x^2 + y^2)^3}$$

$$q = \frac{\mu}{2\pi} \cdot \frac{2xy}{x^2 + y^2}$$

The stream function for a doublet could have been derived by integration in the following manner:

For a source and sink of equal strength

$$u = \frac{\mu}{2\pi} \cdot \frac{x^2 - y^2 - s^2}{(x^2 + y^2 + s^2)^{3/2}} + 4ys^2$$

$$v = \frac{\mu}{2\pi} \cdot \frac{2xy}{(x^2 + y^2 + s^2)^{3/2}} + 4xs^2$$

Let $s \rightarrow 0$. Then the x and y components of velocity approach

$$u = \frac{\mu}{2\pi} \cdot \frac{x^2 - y^2}{(x^2 + y^2)^{3/2}}$$

$$v = \frac{\mu}{2\pi} \cdot \frac{2xy}{(x^2 + y^2)^{3/2}}$$

Now since

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$

$$= \frac{\mu}{2\pi} \cdot \frac{-2xy}{(x^2 + y^2)^{3/2}} dx + \frac{\mu}{2\pi} \cdot \frac{x^2 - y^2}{(x^2 + y^2)^{3/2}} dy$$

$$= \frac{\mu}{2\pi} \left[\frac{(x^2 - y^2) dy - 2xy dx}{(x^2 + y^2)^{3/2}} \right]$$

This last expression can be shown to be an exact differential.

Integration of it gives

$$\psi = \frac{\mu}{2\pi} \cdot \frac{y}{(x^2 + y^2)^{3/2}} + \text{constant}$$

Let $\psi = 0$ where $\begin{cases} x = 0 \\ y = 0 \end{cases}$

Then

$$\psi = \frac{\mu}{2\pi} \cdot \frac{y}{(x^2 + y^2)^{3/2}}$$

This stream function combined with one for a uniform flow in

the negative direction of x leads to a very interesting result.

$$\psi = -Uy + \frac{\mu}{2\pi} \frac{y}{x^2+y^2}$$

Let $\psi = 0$

$$U = \frac{\mu}{2\pi} \frac{1}{x^2+y^2}$$

or

$$x^2+y^2 = \frac{\mu}{2\pi U} \quad \begin{cases} U = \text{constant.} \\ \mu = \text{constant.} \end{cases}$$

Then $x^2+y^2 = \text{constant}$, the Equation of a circle.

Let the radius of this circle $= a$

$$x^2+y^2 = a^2 = \frac{\mu}{2\pi U}$$

or $\mu = 2\pi a^2 U$

Substituting in the Equation for the combined stream function

$$\psi = -Uy + \frac{a^2 U y}{x^2+y^2}$$

$$\psi = -Uy \left(1 - \frac{a^2}{x^2+y^2}\right) \quad \text{Equation 3. Or in polar co-ordi-}$$

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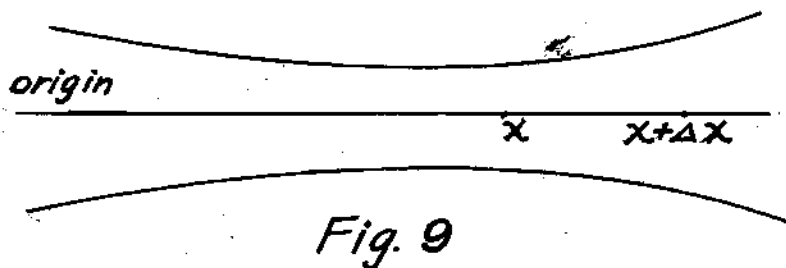
$$\psi = -U \left(r - \frac{a^2}{r}\right) \sin \theta$$

This expression represents the flow about a cylinder of a stream which at a remote distance from the axis of the cylinder is uniform and has a velocity $-U$ parallel to the x axis. The axis of the cylinder passes through the origin of co-ordinates perpendicular to the xy plane and is infinite in length. This flow can not be approximated except in fluids having large internal friction like heavy oil, syrups, etc. In air the flow approximates very closely the ideal one of

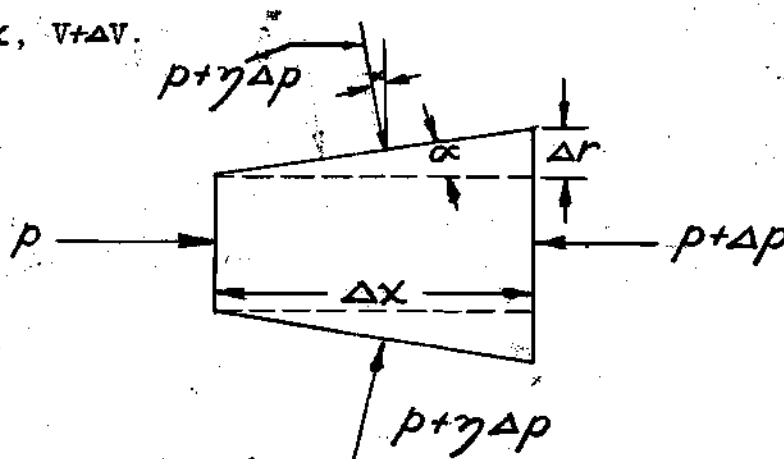
the stream function on the upstream side but it breaks away very quickly if the air velocity is large compared with the radius "a" and leaves a large dead air space on the downstream side. This function, however, is an important one in the theory of the flow of air about an airfoil as will be shown in later discussions. For a comparison of the theoretical flow with the experimental see Fig. I4, page 3I of Glauert's "Aerofoil and Air Screw Theory".

BERNOULLI'S EQUATION

The derivation of Bernoulli's Equation for steady rectilinear stream line flow in which the fluid elements do not receive any acceleration components due to change in direction of their velocities.



For convenience only, assume the stream tube between X and $X + \Delta X$ to be a frustum of a right circular cone. Let the barometric pressure in the fluid at X be p and at $X + \Delta X$, $p + \Delta p$. Let the radius of the stream tube at X be r and at $X + \Delta X$, $r + \Delta r$. Let the mean velocity in the stream tube at X be V and at $X + \Delta X$, $V + \Delta V$.



The average pressure on the sides is taken to be $p + \gamma \Delta p$ where $\gamma \Delta p$ means a fractional part of Δp .

FIG. 10

Inspection of Fig. 10 permits the relation

$$\tan \alpha = \frac{\Delta r}{\Delta x}$$

$$\Delta r = \Delta x \tan \alpha$$

RESULTANT FORCE ON A FLUID ELEMENT.

Total force at $x = \pi r^2 p$

Total force at $x + \Delta x = \pi (r + \Delta x \tan \alpha)^2 (p + \Delta p)$

Component parallel to x of the total force on the lateral

$$\text{area} = \left[\frac{2\pi r + 2\pi (r + \Delta x \tan \alpha)}{2} \right] \cdot \frac{\Delta x}{\cos \alpha} \cdot (p + \frac{1}{2} \Delta p) \cdot \sin \alpha$$

$$= 2\pi r \cdot \Delta x \tan \alpha p + \pi \tan^2 \alpha \cdot p \Delta x^2 + 2\pi r \tan \alpha \Delta p \Delta x + \pi \tan^2 \alpha \frac{1}{2} \Delta p \Delta x^2$$

Resultant force =

$$\begin{aligned} \Delta F &= \cancel{\pi r^2 p} + 2\pi r \tan \alpha \cdot \Delta x p + \pi \tan^2 \alpha \Delta x^2 p + \\ & 2\pi r \tan \alpha \cdot \frac{1}{2} \Delta p \cdot \Delta x - \cancel{\pi r^2 p} - \cancel{2\pi r \tan \alpha \cdot \Delta x p} - \cancel{\pi \tan^2 \alpha \cdot \Delta x^2 p} \\ & - \cancel{\pi r^2 \Delta p} - \cancel{2\pi r \tan \alpha \cdot \Delta p \Delta x} - \pi \tan^2 \alpha \cdot \Delta p \cdot \Delta x^2 + \pi \tan^2 \alpha \frac{1}{2} \Delta p \Delta x^2 \\ \frac{\Delta F}{\Delta x} &= 2\pi r \tan \alpha \cdot \frac{1}{2} \Delta p + \pi \tan^2 \alpha \cdot \frac{1}{2} \Delta p \cdot \Delta x - \pi r^2 \frac{\Delta p}{\Delta x} \\ & - 2\pi r \tan \alpha \cdot \Delta p - \pi \tan^2 \alpha \cdot \Delta p \cdot \Delta x^2 \end{aligned}$$

Let $\Delta x \rightarrow 0$ then $\Delta p \rightarrow 0$ and $\lim_{\Delta x \rightarrow 0} \frac{\Delta p}{\Delta x} = \frac{dp}{dx}$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta F}{\Delta x} = \frac{dF}{dx} = -\pi r^2 \frac{dp}{dx}$$

Let $\pi r^2 = S$

Then $dF = -S dp$

ELEMENTARY MASS.

Δ (Volume) = volume of a cylinder of radius r and length ΔX
 + volume of portion of frustum of cone due to r increasing to $r + \Delta r$ as X changes from X to $X + \Delta X$.

The latter Volume by the theory of Pappus is equal to the area of the triangle ($\frac{\Delta X \cdot \Delta X \tan \alpha}{2}$) multiplied by the circumference of a circle whose radius is the distance from the X axis to the centroid of the triangle.

$$\begin{aligned}\therefore \Delta(\text{Vol}) &= \pi r^2 \Delta X + 2\pi(r + \Delta X \tan \alpha) \cdot \left(\frac{\Delta X^2 \tan \alpha}{2}\right) \\ &= \pi r^2 \Delta X + \pi r \tan \alpha \cdot \Delta X^2 + \pi \tan^2 \alpha \cdot \frac{\Delta X^3}{3} \\ \frac{\Delta(\text{Vol})}{\Delta X} &= \pi r^2 + \pi r \tan \alpha \cdot \Delta X + \frac{\pi \tan^2 \alpha \cdot \Delta X^2}{3}\end{aligned}$$

Let $\Delta X \rightarrow 0$

$$\lim_{\Delta X \rightarrow 0} \frac{\Delta(\text{Vol})}{\Delta X} = \frac{d(\text{Vol})}{dX} = \pi r^2$$

$$\therefore d(\text{Vol}) = \pi r^2 dx$$

If ρ = mass density of fluid

$$d(\text{mass}) = \rho d(\text{Vol}) = \rho \pi r^2 dx$$

Let $\pi r^2 = S$

$$d(\text{mass}) = \rho S dx$$

ACCELERATION.

It is now necessary to get an expression for the acceleration of the fluid element at X .

If the motion were not steady an observer at a point in the plane of motion would note that the velocity of a particle as

it reached the point would be different from the velocity of the particle which preceded it. In other words the velocity would not only be different with a change of position of an observer in the plane of motion but would also be different with a change of time.

In mathematical language the velocity of a particle is a function of position and time. So we write that for a perfectly general motion.

$$\text{Velocity} = V = F(x, t)$$

$$\text{and } dV = \frac{\partial V}{\partial x} \cdot dx + \frac{\partial V}{\partial t} \cdot dt$$

$$\text{but acceleration} = \frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial t}$$

$$\text{and } \frac{dx}{dt} = V (\text{The velocity of particle at } x)$$

However we assumed that V could only vary due to a change in x (a change in position) when we set out to derive Bernoulli's Equation for steady motion.

Therefore,

$$\frac{\partial V}{\partial t} = 0$$

$$\frac{\partial V}{\partial x} = \frac{dV}{dx}$$

and $\frac{dV}{dt} = V \cdot \frac{dV}{dx}$

Assembling our expressions for mass, acceleration and force

in $dF = dM \cdot \frac{dV}{dt}$

$$\rho S dx \cdot V \frac{dV}{dx} = -S dp$$

or $dp + V dV = 0$

Now for an incompressible fluid, as assumed, ρ will not vary and we have an Equation we can integrate.

Thus obtaining

$$p + \frac{1}{2}\rho V^2 = \text{constant} = H. \quad (4)$$

This is Bernoulli's Equation for steady flow of an incompressible fluid. It was derived on the assumption that there were no components of acceleration perpendicular to the velocity. The Equation is independent of x and it may be found applicable for flows which have curvilinear stream tubes. In fact it will be shown that its application is not as restricted as the assumptions would indicate.

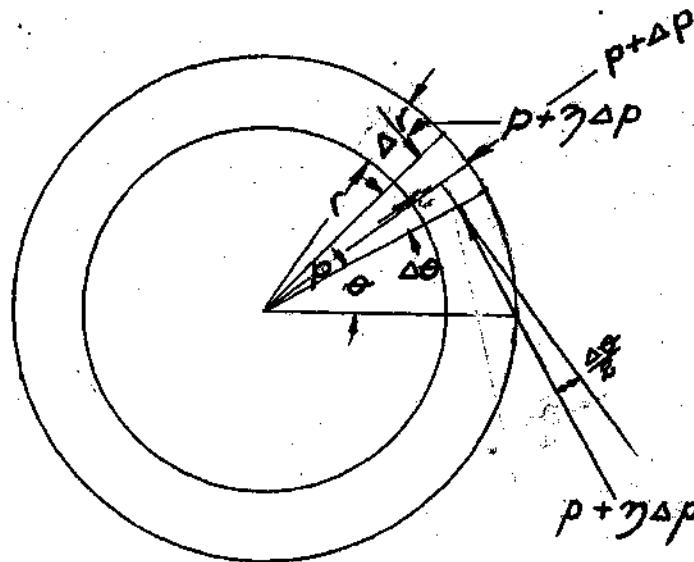
BERNOULLI'S EQUATION FOR A FLOW IN WHICH THE STREAM LINES ARE CIRCLES.

This is the simplest case of a flow in which the fluid elements receive components of acceleration due to a change in direction of their velocities. Furthermore, such a motion occurs in air in the form of whirlwinds, cyclones, etc. We know that such a motion could not be started in a fluid without viscosity and therefore we will assume enough viscosity in the air to get the motion started. Once started we will neglect the influence of viscosity. There are many kinds of air motion where the effect of viscosity is negligible from a practical standpoint and this type of motion comes under such a classification except in regions of excessive velocity or of very great changes of velocity along a line perpendicular to the streamlines.

If the motion is steady and the streamlines are circles the magnitude of the velocity at every point on the same circle must be constant for if not the fluid would have to be compressible

which is contrary to our basic assumption, incompressibility.

Fig. 11



ELEMENTARY MASS.

$$\text{Volume of ring} = [\pi(r + \Delta r)^2 - \pi r^2] \times 1 \text{ ft.} = \pi(2r\Delta r + \Delta r^2) \times 1 \text{ ft.}$$

$$\text{Volume per radian} = \frac{\pi(2r\Delta r + \Delta r^2)}{2\pi} = r\Delta r + \frac{\Delta r^2}{2}$$

$$\text{Volume for } \Delta\theta \text{ part of a radian} = r\Delta r\Delta\theta + \frac{\Delta r^2\Delta\theta}{2}$$

$$\Delta(\text{mass}) = \rho r \Delta r \Delta\theta + \rho \frac{\Delta r^2 \Delta\theta}{2}$$

$$\frac{\Delta(\text{mass})}{\Delta r \Delta\theta} = \rho r + \frac{\rho \Delta r}{2}$$

$$\lim_{\substack{\Delta\theta \rightarrow 0 \\ \Delta r \rightarrow 0}} \frac{\Delta(\text{mass})}{\Delta r \Delta\theta} = \rho r$$

$$\therefore d(\text{mass}) = \rho r dr d\theta$$

RESULTANT FORCE ON AN ELEMENT.

Let Resultant force on mass $[p r \Delta r \Delta \theta + \frac{\rho \Delta r^2 \Delta \theta}{2}] = \Delta F$

$$\Delta F = p \cdot [\overset{\text{Area}}{r \Delta \theta \times 1 \text{ ft.}}] + 2(p + \gamma \Delta p) \cdot [\overset{\text{Area}}{\Delta r \times 1 \text{ ft.}}] \cdot \sin \frac{\Delta \theta}{2} - (p + \Delta p) \times [\overset{\text{Area}}{(r + \Delta r) \Delta \theta \times 1 \text{ ft.}}] = p r \Delta \theta + 2 p \cdot \Delta r \cdot \sin \frac{\Delta \theta}{2} + 2 \gamma \Delta p \Delta r \sin \frac{\Delta \theta}{2} - p r \Delta \theta - r \Delta p \Delta \theta - p \Delta r \Delta \theta - \Delta p \Delta r \Delta \theta$$

$$\text{then } \frac{\Delta F}{\Delta r \Delta \theta} = 2 p \cdot \frac{\sin \frac{\Delta \theta}{2}}{\Delta \theta} + 2 \gamma \Delta p \frac{\sin \frac{\Delta \theta}{2}}{\Delta \theta} - r \frac{\Delta p}{\Delta r} - p - \Delta p$$

$$\text{as } \Delta \theta \rightarrow 0 \quad \lim_{\Delta \theta \rightarrow 0} \frac{\sin \frac{\Delta \theta}{2}}{\Delta \theta} = \frac{1}{2} \quad \text{also as } \Delta r \rightarrow 0 \quad \Delta p \rightarrow 0$$

$$\text{as } \Delta r \rightarrow 0 \quad \lim_{\Delta r \rightarrow 0} \frac{\Delta p}{\Delta r} = \frac{dp}{dr}$$

$$\lim_{\substack{\Delta r \rightarrow 0 \\ \Delta \theta \rightarrow 0}} \frac{\Delta F}{\Delta r \Delta \theta} = p - r \frac{dp}{dr} - p = -r \frac{dp}{dr}$$

$$\therefore dF = -r \frac{dp}{dr} \cdot dr d\theta = -r dp d\theta$$

Note: For a perfectly general motion the $\lim_{\Delta r \rightarrow 0} \frac{\Delta p}{\Delta r}$ would have been $\frac{dp}{dr}$ but for our discussion p does not vary with θ and

$$\lim_{\Delta r \rightarrow 0} \frac{\Delta p}{\Delta r} = \frac{dp}{dr}$$

ACCELERATION OF AN ELEMENT.

The only component of acceleration for the assumed motion is one toward the center of the circle due to a change in direction of the velocity.

$$\therefore \text{acceleration} = - \frac{V'^2}{r}$$

Since the resultant force along any line is equal to the mass upon which the force acts multiplied by the component parallel to that line of the acceleration of the mass center

$$-r dp d\theta = p r dr d\theta \left(- \frac{V'^2}{r} \right)$$

$$\therefore dp = \rho V'^2 \frac{dr}{r}$$

This Equation is in agreement with the original assumptions for inspection shows that no variable is a function of θ .

Bernoulli's Equation was derived for changes in the magnitude of the velocity along a stream line. There is no change in magnitude of velocity or pressure along the assumed circular stream lines. If any change occurs in either the pressure or the magnitude of the velocity it will be along a line (the radius in this case) perpendicular to the stream lines.

It is common observation that in whirlwinds, cyclones, etc. the velocity decreases as the radius from the center of the disturbance increases. We have no way of knowing the exact law of variation of the velocity with the radius except by experiment. All we know is that it decreases as the radius increases. A simple law of variation would be

$$V' = \frac{C}{r}$$

where C = constant.

This would make the magnitude of the velocity inversely proportional to the radius. While it may not be this way in nature, it constitutes a good first assumption.

Substituting $\frac{C}{V'}$ for r and $-\frac{Cdr}{V'^2}$ for dr in

$$dp = \rho \frac{V'^2}{r} \cdot dr$$

we get

$$dp = \rho \cdot \frac{V'^2}{\frac{C}{V'}} \cdot \left(-\frac{Cdr}{V'^2} \right) = -\rho V' dV'$$

or

$$dp + \rho V' dV' = 0$$

Integrating

$$p + \frac{1}{2} \rho V'^2 = \text{constant} = H$$

Therefore if we have a circular stream line flow of an incompressible, non-viscous fluid in which $v/r = c$ (a constant), Bernoulli's Equation expresses the relationship between the pressure at any point and the velocity at that point although changes in the magnitude of the velocity can only occur along a line perpendicular to the stream line.

From all of the preceding discussion it is obvious that if the variation of the magnitude of the velocity with the radius is according to any other law than $v/r = c$ the quantity H can not be a constant but must also vary with the radius.

AN INTERPRETATION OF BERNOULLI'S EQUATION.

Bernoulli's Equation has been derived by considering changes in vector quantities. It is evident that the result does not express a relationship between vector quantities. Consider the units of the terms: P is in pounds per square foot but it could just as well be written.

$$\frac{\text{lbs}}{\text{ft}^2} \times \frac{\text{ft}}{\text{ft}} = \frac{\text{ft} \cdot \text{lbs}}{\text{ft}^3}$$

The units of ρv^2 are

$$\rho \left(\frac{\text{lbs} \cdot \text{sec}^2}{\text{ft}^4} \right) \times v^2 \left(\frac{\text{ft}^2}{\text{sec}^2} \right) = \rho v^2 \left(\frac{\text{ft} \cdot \text{lbs}}{\text{ft}^3} \right)$$

The units of each term, therefore, can represent energy per cubic foot of the fluid. The pressure term will represent a potential energy and the term $\frac{1}{2} \rho v^2$ will represent kinetic energy. On this basis Bernoulli's Equation is applicable to

any flow in which there are no losses or gains in total energy. Therefore if q is the resultant velocity at any point of such a flow

$$p + \frac{1}{2}\rho q^2 = H$$

CIRCULATION

In a steady flow where the stream lines are circles $Vr = c$.

Since c is a constant it could be taken equal to $\frac{K}{2\pi}$. Making this substitution

$$2\pi r \cdot V = K$$

K is called the circulation. It seems to be a rather artificially defined quantity in view of the difficulty one has of grasping its physical significance. In spite of this it is a very useful quantity and should be thoroughly understood. Suppose that an observer followed a closed curve always keeping his left hand on the inside of the curve and that the motion was

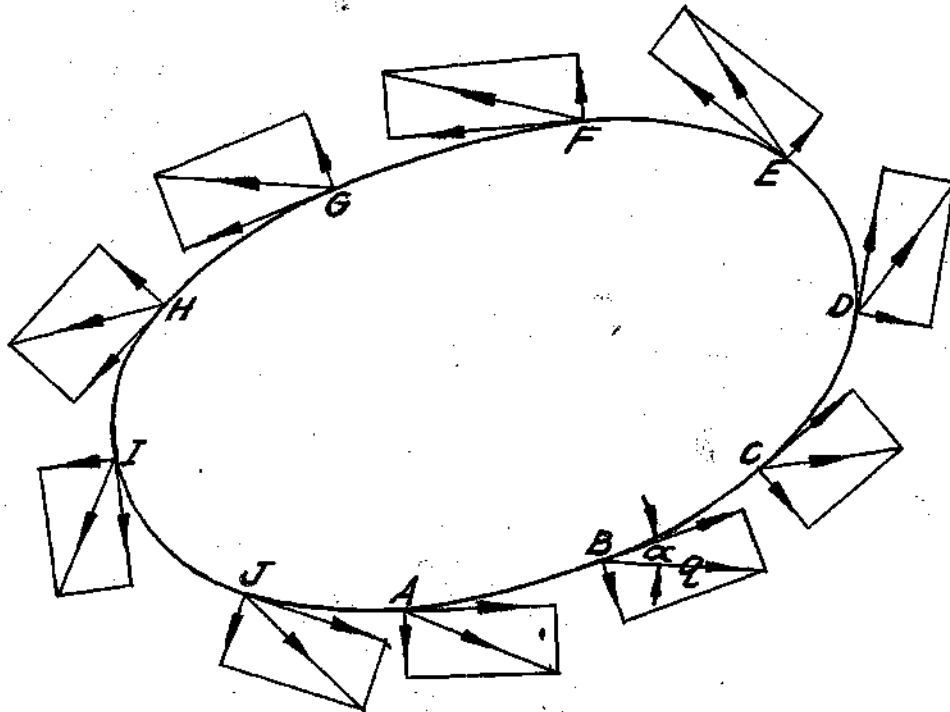


Fig. 12

such that at every point of the curve the fluid had a component of velocity along the tangent to the curve with the same sense as the motion of the observer, then the observer would conclude that the fluid was "circulating" around the closed curve. But this is only a qualitative conception of it. Construct a curve with an abscissa equal to the distance from the starting point to any point on the closed curve and with an ordinate equal to the tangential component of the velocity at that point. The area, to the proper scale, between this curve and the abscissa axis is the circulation. This would be equivalent to building a picket fence around the curve making each picket perpendicular to the plane of motion and of a length equal to the tangential component of the velocity at the picket. The area of this picket fence would be the circulation around the closed curve. In constructing the picket fence the builder must travel around the curve, facing the way he is going, keeping his left arm on the inside of the curve and he must consider a picket positive if the tangential component of velocity which it represents has the same sense as his direction of motion. All tangential components will not necessarily be of the same sign. If there is more positive than negative area the circulation is positive and vice-versa.

Around the wing of an airplane the circulation is made up of one sign of area above the wing and the other sign below the wing but the area above is greater. For the flow which has circular stream lines each picket is V' high and the fence is $2\pi r$ long and the area of the fence is $2\pi r V'$. All this leads up to the mathematical statement of the circulation.

$$K = \int q \cos \alpha \, ds$$

See Fig. 12.

q = resultant velocity at any point.

α = angle between line of q and the tangent to curve.

ds = elementary length of arc at the point.

The integral is to be taken all the way around the curve in the positive direction and is called a line integral.

The stream function for the case with circular stream lines.

From Equation 2

$$-\frac{\partial \psi}{\partial r} = V'$$

$$\frac{\partial \psi}{\partial \theta} = 0$$

But $V' = \frac{K}{2\pi r}$

and substituting these relations in

$$d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial r} dr$$

$$d\psi = -V' dr = -\frac{K}{2\pi} \frac{dr}{r}$$

Integrating

$$\psi = -\frac{K}{2\pi} \log r + \text{constant}$$

Let $\psi = 0$ where $r = a$

$$0 = -\frac{K}{2\pi} \log a + \text{constant}$$

Then

$$\psi = -\frac{K}{2\pi} \log r + \frac{K}{2\pi} \log a = -\frac{K}{2\pi} \log \frac{r}{a}$$

For the special case where $a = \text{unity}$

$$\psi = -\frac{K}{2\pi} \log r.$$

RESULTANT FORCE OF A FLUID IN MOTION.

The computation of the resultant force on a cylinder of radius "a" of a perfect fluid with the following stream function

$$\psi = Uy \left(1 - \frac{a^2}{r^2}\right) = -U \left(r - \frac{a^2}{r}\right) \sin \theta$$

is zero.

The polar form of the above equation is the best for demonstrating this.

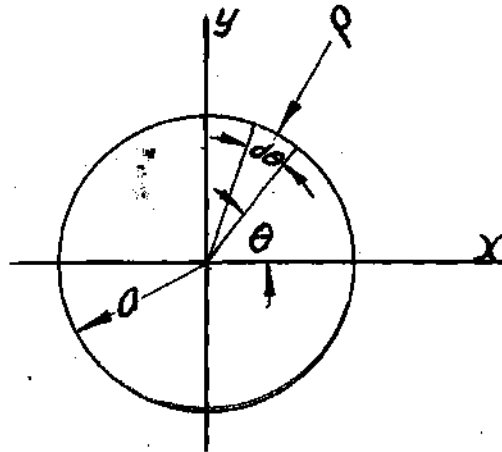


Fig. 13.

Since the stream function is the same for every foot of the cylinder of infinite length, the force on every foot will be the same and it will only be necessary to consider one foot of its total length.

Let p = barometric pressure at any point on cylinder.

Let elementary area = $a d\theta \times 1 \text{ ft.}$

Let dF = total force on elementary area = $p a d\theta$

Then $(dF)_x$ = Horizontal component of $dF = p a \cos \theta d\theta$

and $(dF)_y$ = Vertical component of $dF = p a \sin \theta d\theta$

Now

$$F_x = \int_0^{2\pi} p a \cos \theta d\theta$$

$p + \frac{1}{2} \rho q^2 = H$ is applicable to this case since there is nothing which can increase or decrease the total energy, the fluid being non-viscous. The Equation, however, must be taken in the pressure sense.

$$p = H - \frac{1}{2} \rho q^2$$

and

$$F_x = \int_0^{2\pi} H a \cos \theta d\theta - \frac{1}{2} \rho a \int_0^{2\pi} q^2 \cos \theta d\theta$$

$$\frac{\partial \psi}{\partial \theta} = u' r = -U \left(r - \frac{a^2}{r} \right) \cos \theta$$

$$-\frac{\partial \psi}{\partial r} = v' = U \left(1 + \frac{a^2}{r^2} \right) \sin \theta$$

And for $r = a$

$$u' = -U \left(\frac{a}{a} - \frac{a^2}{a^2} \right) \cos \theta = 0$$

$$v' = U \left(1 + \frac{a^2}{a^2} \right) \sin \theta = 2U \sin \theta$$

$$q = \sqrt{u'^2 + v'^2} = \sqrt{0^2 + 2U \sin \theta^2} = 2U \sin \theta$$

Remembering that H , U , and a are not variables for this integration

$$F_x = H a \int_0^{2\pi} \cos \theta d\theta - 2U^2 \rho a \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta$$

$$= \left[H a \sin \theta - 2U^2 \rho a \frac{\sin^3 \theta}{3} \right]_0^{2\pi} = 0$$

$$\begin{aligned}
 F_y &= - \int_0^{2\pi} p a \sin \theta \, d\theta = \int_0^{2\pi} H a \sin \theta \, d\theta - \frac{1}{2} \rho a \int_0^{2\pi} q^2 \sin \theta \, d\theta \\
 &= -H a \int_0^{2\pi} \sin \theta \, d\theta + 2U^2 \rho a \int_0^{2\pi} \sin^3 \theta \, d\theta
 \end{aligned}$$

But $\sin^3 \theta = (1 - \cos^2 \theta) \sin \theta = \sin \theta - \cos^2 \theta \sin \theta$

$$\begin{aligned}
 F_y &= \left[H a \cos \theta - 2U^2 \rho a \cos \theta + \frac{2U^2 \rho a \cos^3 \theta}{3} \right]_0^{2\pi} \\
 &= 0
 \end{aligned}$$

Therefore, a perfect fluid having the assumed stream function can not exert any force on the submerged cylinder other than that due to buoyancy. A simple inspection of

$$p = H - \frac{1}{2} \rho q^2 = H - 2\rho U^2 \sin^2 \theta$$

would have shown that no resultant force could be obtained.

H is constant and therefore for every pressure of H at the end of a diameter there is an equal and opposite one at the other end of the same diameter. This being true for every diameter, the part of p dependent on H can not produce a resultant force. The identical reasoning can be applied to the

$2\rho U^2 \sin^2 \theta$ term.

LIFT ON A CYLINDER SUBMERGED IN A FLUID.

If a stream function could be found which when superposed

upon

$$\psi = -U\left(r - \frac{a^2}{r}\right)\sin\theta$$

the symmetrical nature of the flow could be changed, then we might expect a force. In the early days of Aeronautical science Lanchester predicted that the nature of this flow would have to be circulatory. Adding the stream function for circular stream line flow to the above one we get

$$\psi = -U\left(r - \frac{a^2}{r}\right)\sin\theta - \frac{K}{2\pi} \log \frac{r}{a}$$

$$\frac{\partial\psi}{\partial\theta} = u'r = -U\left(r - \frac{a^2}{r}\right)\cos\theta$$

$$-\frac{\partial\psi}{\partial r} = v' = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{K}{2\pi r}$$

and on the cylinder where $r=a$

$$u'=0$$

$$v' = 2U\sin\theta + \frac{K}{2\pi a}$$

Substituting in

$$p = H - \frac{1}{2}\rho q^2$$

$$p = H - \frac{1}{2}\rho\left(4U^2\sin^2\theta + \frac{2UK\sin\theta}{\pi a} + \frac{K^2}{4\pi^2 a^2}\right)$$

The H term being constant can not contribute to a resultant force so we will ignore it. Then

$$F_x = -\frac{1}{2}\rho a \int_0^{2\pi} \left(4U^2\sin^2\theta + \frac{2UK\sin\theta}{\pi a} + \frac{K^2}{4\pi^2 a^2}\right) \cos\theta d\theta$$

$$2\rho U^2 \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta = 0 \quad \left[\text{from a preceding integration} \right]$$

$$\frac{K^2}{4\pi^2 a} \int_0^{2\pi} \cos \theta d\theta = 0. \quad \text{ditto.}$$

$$\frac{2UK}{\pi a} \int_0^{2\pi} \sin \theta \cos \theta d\theta = \frac{2UK}{\pi a} \cdot \frac{\sin^2 \theta}{2} \Big|_0^{2\pi} = 0$$

$$\therefore F_x = 0$$

$$F_y = + \frac{1}{2} \rho a \int_0^{2\pi} \left(4U^2 \sin^2 \theta + \frac{2UK \sin \theta}{\pi a} + \frac{K^2}{4\pi^2 a^2} \right) \sin \theta d\theta$$

$$2\rho U^2 \int_0^{2\pi} \sin^3 \theta d\theta = 0 \quad \left[\text{from a preceding integration} \right]$$

$$\frac{2UK}{\pi a} \int_0^{2\pi} \sin^2 \theta d\theta = \frac{2UK}{\pi a} \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{2UK}{\pi a} \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi}$$

$$= \frac{2UK\pi}{\pi a}$$

$$\therefore F_y = \frac{1}{2} \rho a \cdot \frac{2UK}{a} = \rho UK$$

For b ft. of a cylinder $F_y = \rho UKb$.

This being a positive y force it must represent a lift. This is a very important fact and is the basis for theoretical determinations of lift from an airfoil. A theory will be presented where-by the circle of radius "a" can be transformed into the shape of an airfoil and the flow around the cylinder can be transformed into the flow about the airfoil in such a manner that the magnitudes of the velocities at all points of the airfoil surface can be computed. Knowing these velocities, the pressure at any point on the airfoil can be calculated by application of Bernoulli's Equation. With this information a pressure distribution curve can be plotted. Moreover, for small angles of attack experimental pressure distribution curves check the theory.

VORTICITY.

In the case of the flow

$$v'r = c$$

or

$$2\pi r v' = K$$

the circle about the origin can be taken with a radius of Δr .

$$K = 2\pi \cdot \Delta r \cdot v' = \pi \cdot \Delta r^2 \cdot \frac{2v'}{\Delta r}$$

If the fluid in the volume $\pi \cdot \Delta r^2 \cdot 1'$ was solid then it would be rotating and its surface velocity, v' , divided by its radius, Δr , would be its angular velocity. If Δr is a finite measurable distance for the actual fluid v' must get larger as Δr decreases $\left[v' = \frac{K}{2\pi \cdot \Delta r} \right]$. Where $\Delta r = \frac{\Delta r}{2}$, $v' = \frac{K}{\pi \Delta r}$. Which shows

that if the radius is made half as much, the velocity is doubled. Now this is exactly opposite to what happens in pure rotation in which the velocity varies directly with the radius $[v = \omega r]$. But if Δr be decreased to an infinitesimal distance then in the limit $\frac{v}{\Delta r}$ can be and is taken as the average angular velocity of the infinitesimal volume $\pi \cdot \overline{dr}^2 \cdot l$.

Let $\pi \overline{dr}^2 = ds$, then

$$K = ds \cdot 2\omega$$

The quantity 2ω is given the name of vorticity.

Suppose that the closed figure about any point is taken to be a rectangle and the velocity at the center of the rectangle has the two rectangular components, u and v .

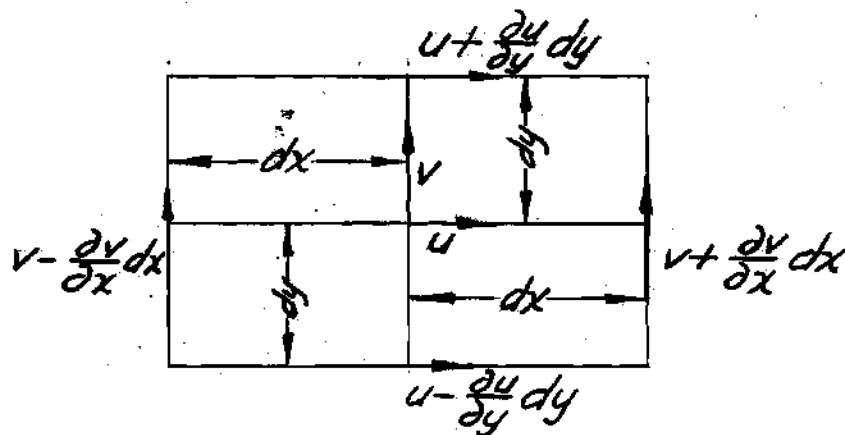


Fig. 14..

For this figure according to the definition of circulation

$$K = \left[u - \frac{\partial u}{\partial y} dy \right] 2dx + \left[v + \frac{\partial v}{\partial x} dx \right] 2dy \\ - \left[u + \frac{\partial u}{\partial y} dy \right] 2dx - \left[v - \frac{\partial v}{\partial x} dx \right] 2dy$$

$$\therefore K = \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] 4 dy dx$$

Because $ds = 4dydx$

and $K = 2\omega ds$

$$2\omega = \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] = 2 \times [\text{average angular velocity of the element.}]$$

but $\frac{\partial \psi}{\partial y} = u$

$$-\frac{\partial \psi}{\partial x} = v$$

so $\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial u}{\partial y}$

$$-\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial v}{\partial x}$$

$$\text{Hence } 2\omega = - \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right]$$

If at a point in the fluid

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

we will know that the vorticity at the point is zero and that there is no circulation about the point.

For a flow the stream function of which is

$$\psi = -\frac{K}{2\pi} \log r = -\frac{K}{4\pi} \log (x^2 + y^2)$$

$$\frac{\partial \psi}{\partial y} = -\frac{K}{4\pi} \frac{2y}{x^2 + y^2}$$

$$\frac{\partial \psi}{\partial x} = -\frac{K}{4\pi} \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2 \psi}{\partial y^2} = -\frac{K}{2\pi} \left[\frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \right] = -\frac{K}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{K}{2\pi} \left[\frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} \right] = -\frac{K}{2\pi} \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{K}{2\pi} \left[\frac{x^2 - y^2 + y^2 - x^2}{(x^2 + y^2)^2} \right]$$

For every point in the plane except the origin

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

i.e. no vorticity.

At the origin

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

[indeterminate.]

But on any circle with its center at the origin

$$K = 2\pi r V' = \pi r^2 2\omega$$

At the origin therefore

$$2\omega = \frac{K}{\Delta r^2 \pi} = -\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

A MORE INVOLVED ILLUSTRATION OF VORTICITY.

Suppose that we had assumed that $v r^2 = C$ when we were discussing circular stream line flow. In such a case

$$-\frac{\partial \psi}{\partial r} = v' = \frac{C}{r^2}$$

$$\frac{\partial \psi}{\partial \theta} = u' r = 0$$

$$-\frac{\partial \psi}{\partial r} dr = \frac{C}{r^2} dr$$

$$+ \psi = + \frac{C}{r} + \text{constant.}$$

Assume $\psi = 0$ where $r = \infty$, then the constant will be zero and

$$\psi = \frac{C}{r}$$

The expression for vorticity is rather involved for this stream function if we attempt to get an expression for it by use of

$$2\omega = -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

The expression will be simplified if we derive and use the polar form.

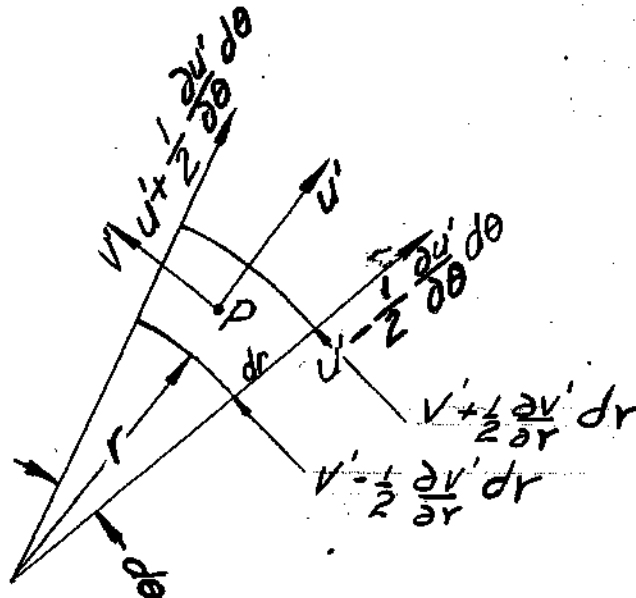


Fig. 15.

Let u' and v' be the components of velocity at P.

$$K = \left[u' - \frac{1}{2} \frac{\partial u'}{\partial \theta} d\theta \right] dr + \left[v' + \frac{1}{2} \frac{\partial v'}{\partial r} dr \right] [r + dr] d\theta \\ - \left[u' + \frac{1}{2} \frac{\partial u'}{\partial \theta} d\theta \right] dr - \left[v' - \frac{1}{2} \frac{\partial v'}{\partial r} dr \right] r d\theta$$

$$K = \cancel{u' dr} - \frac{1}{2} \frac{\partial u'}{\partial \theta} dr d\theta + \cancel{v' r d\theta} + v' dr d\theta + \frac{1}{2} \frac{\partial v'}{\partial r} r dr d\theta \\ + \frac{1}{2} \frac{\partial v'}{\partial r} \cdot dr^2 d\theta - \cancel{u' dr} - \frac{1}{2} \frac{\partial u'}{\partial \theta} dr d\theta - \cancel{v' r d\theta} + \frac{1}{2} \frac{\partial v'}{\partial r} r dr d\theta \\ = \frac{\partial v'}{\partial r} \cdot r dr d\theta + v' dr d\theta - \frac{\partial u'}{\partial \theta} dr d\theta + \frac{1}{2} \frac{\partial v'}{\partial r} \cdot dr^2 d\theta \\ = \left[\frac{\partial v'}{\partial r} - \frac{\partial u'}{r \partial \theta} + \frac{v'}{r} + \frac{1}{2} \cdot \frac{1}{r} \frac{\partial v'}{\partial r} dr \right] r dr d\theta$$

Clearly $\frac{1}{2r} \cdot \frac{\partial v'}{\partial r} dr$ is negligible or in other words in the preceding expression $\frac{1}{2} \frac{\partial v'}{\partial r} dr^2 d\theta$ is of higher order than any of the other terms in that expression, and therefore may be omitted.

But

$$\frac{\partial \psi}{\partial \theta} = u'r$$

$$-\frac{\partial \psi}{\partial r} = v'$$

$$\therefore \frac{1}{r} \cdot \frac{\partial^2 \psi}{\partial \theta^2} = \frac{\partial u'}{\partial \theta}$$

$$-\frac{\partial^2 \psi}{\partial r^2} = \frac{\partial v'}{\partial r}$$

so

$$K = - \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right] r d\theta dr = 2\omega \cdot dS$$

But $dS = r d\theta dr$.

$$\therefore -2\omega = \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right]$$

But for the flow under consideration

$$\psi = \frac{c}{r}$$

so $\frac{\partial \psi}{\partial \theta} = 0$

$$\frac{\partial \psi}{\partial r} = -\frac{c}{r^2}$$

and

$$\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

$$\frac{\partial^2 \psi}{\partial r^2} = +\frac{2c}{r^3}$$

and,

$$\text{Vorticity at any point} = -2\omega = \left(\frac{2c}{r^3} - \frac{c}{r^3} \right) = \frac{c}{r^3}$$

$$\text{or } 2\omega = -\frac{c}{r^3}$$

This means that there is vorticity at every point in the plane

and not just at the origin as in the case where

$$\psi = -\frac{K}{2\pi} \log r.$$

MECHANISM OF VORTICITY.

Because an element of fluid whose volume is a cylinder of radius dr with an axis 1 ft. long and passing through any point B in the plane is so closely analagous to a roller bearing it is worth while to make the effort to understand it. The velocity of the point B of the fluid is $\frac{C}{r^2}$. The velocity of

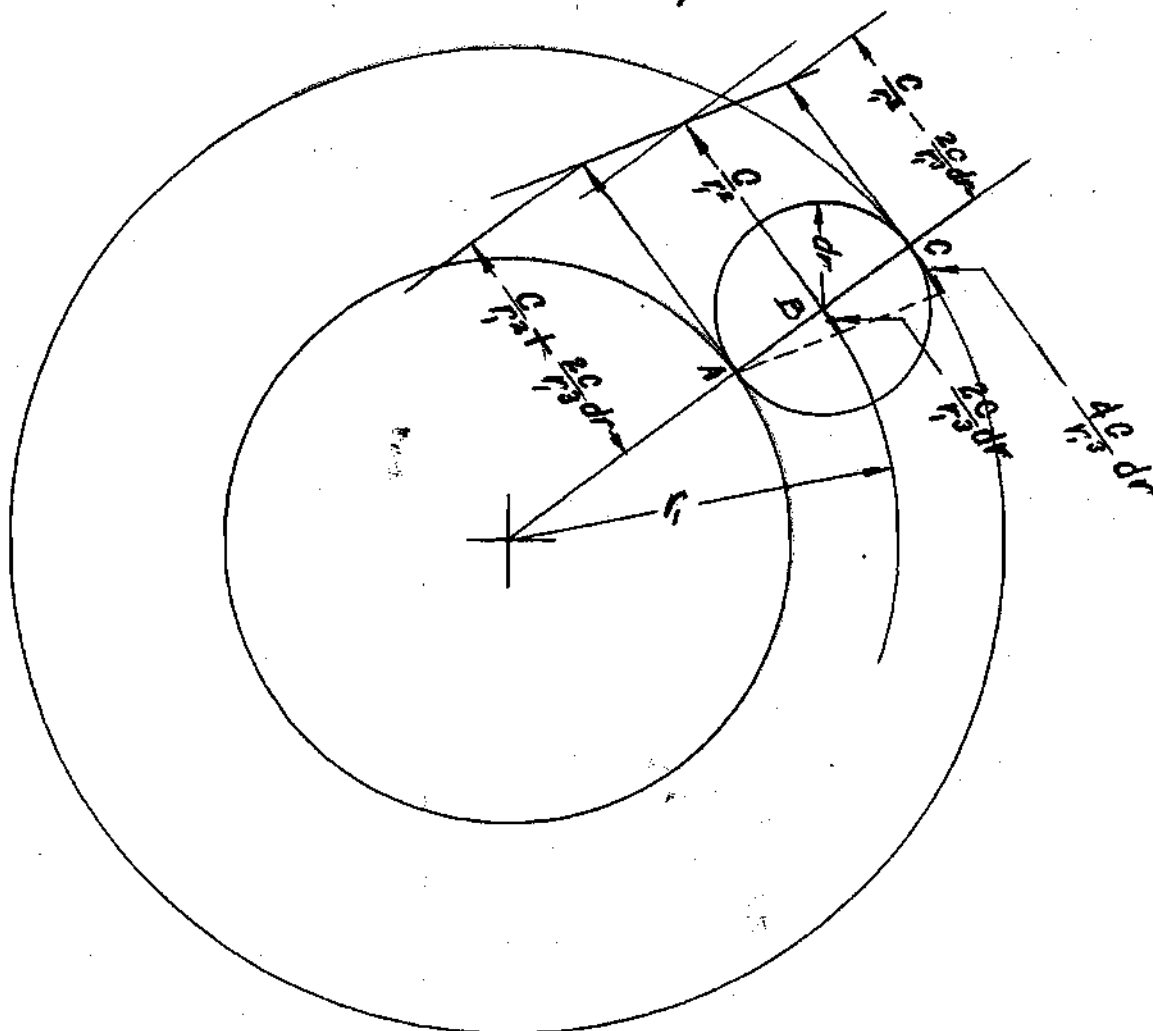


Fig. 16.

point C is

$$V_c = V_b + \left(\frac{\partial V}{\partial r}\right)_B dr$$

$$V_c = \frac{C}{r^2} - \frac{2Cdr}{r^3}$$

and of point A is

$$V_A = \frac{C}{r^2} + \frac{2Cdr}{r^3}$$

We will next determine the relative velocities of B and C with respect to A. Since two points will continue to have the same relative velocity if the same velocity is subtracted from each we can get the relative velocities of B and C with respect to A if the velocity of A is subtracted from A, B, and C for this makes A stand still and whatever velocities B & C have after subtraction will be their velocities relative to A. But this gives a backward velocity to B of $\frac{2C}{r^3} dr$ and a backward velocity to C of $\frac{4C}{r^3} dr$. But these are the identical relative velocities the same points on a roller bearing would have with respect to a shaft whose outside radius was that of the fluid of radius $(r-dr)$. The roller bearing would serve to step down the velocity from that of the shaft to that of the sleeve of radius $(r+dr)$ just as the circulating fluid around point B steps down the velocity in the fluid from $\frac{C}{r^2} + \frac{2C}{r^3} dr$ at A to $\frac{C}{r^2} - \frac{2C}{r^3} dr$ at C. Hence all the fluid contained in the ring between the radii $(r,-dr)$ and $(r,+dr)$ acts like a set of roller bearings would act between a shaft of radius $(r,-dr)$ and a sleeve of inside radius $r,+dr$. Since r is any radius this explanation applies to any point in the entire plane. The full significance of this can best

be understood after a further study of Bernoulli's Equation.

VARIATIONS OF H IN BERNOULLI'S EQUATION.

Let s = distances measured along a stream line.

and r = distances measured perpendicular to a stream line.

$$\rho + \frac{1}{2} \rho q^2 = H$$

Now

$$\frac{\partial \rho}{\partial s} + \rho q \frac{\partial q}{\partial s} = \frac{\partial H}{\partial s}$$

But along a stream line of a non-viscous, incompressible fluid the quantity H in Bernoulli's Equation can not vary unless energy is added or subtracted in some way.

In general, therefore

$$\frac{\partial \rho}{\partial s} + \rho q \frac{\partial q}{\partial s} = 0$$

Similarly

$$\frac{\partial \rho}{\partial r} + \rho q \frac{\partial q}{\partial r} = \frac{\partial H}{\partial r}$$

But along a line perpendicular to a stream line

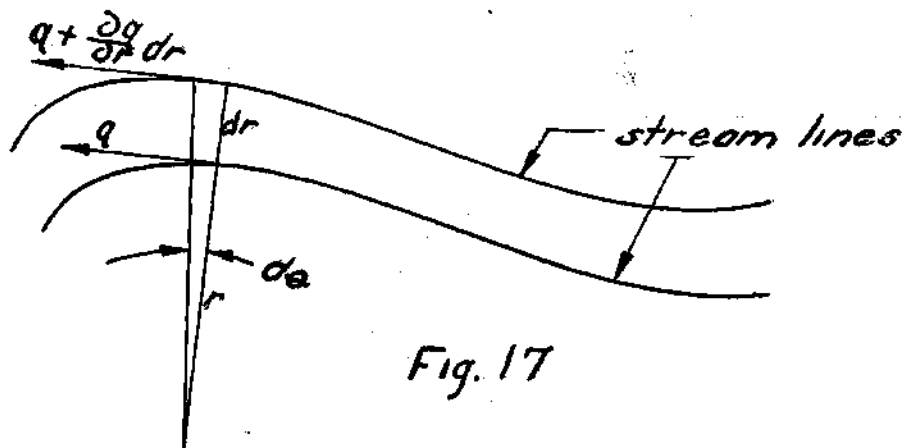
$$\frac{\partial \rho}{\partial r} = \rho \frac{q^2}{r}$$

so

$$\rho \frac{q^2}{r} + \rho q \frac{\partial q}{\partial r} = \frac{\partial H}{\partial r}$$

or

$$\rho q \left(\frac{q}{r} + \frac{\partial q}{\partial r} \right) = \frac{\partial H}{\partial r}$$



Circulation around the elementary area is

$$K = [q + \frac{\partial q}{\partial r} dr](r+dr)d\theta - qr d\theta$$

And neglecting infinitesimals of higher order

$$K = \frac{\partial q}{\partial r} r dr d\theta + qr d\theta \\ = [\frac{\partial q}{\partial r} + \frac{q}{r}] r dr d\theta$$

There can be no component of velocity perpendicular to the stream lines, so

$$\frac{\partial \psi}{\partial \theta} = r \frac{\partial u'}{\partial r} = 0$$

$$\therefore \frac{\partial q}{\partial r} + \frac{q}{r} = 2\omega = \text{vorticity}$$

substituting 2ω for $\frac{\partial q}{\partial r} + \frac{q}{r}$ in $\frac{\partial H}{\partial r}$

$$\frac{\partial H}{\partial r} = \rho q \cdot 2\omega$$

Therefore, changes in H along a line perpendicular to stream lines are produced by vorticity.

Returning to the discussion of the flow for which $\psi = \frac{C}{r}$ we are able to conclude that Bernoulli's Equation does not apply because for this case

$$2\omega = -\frac{C}{r^2}$$

and since

$$\frac{\partial H}{\partial r} = 2\omega \rho q$$

$$\frac{\partial H}{\partial r} = -\frac{C}{r} \cdot \rho v'$$

The only possible way according to our assumptions H can

change is with respect to r ; so, $\frac{\partial H}{\partial r}$ becomes $\frac{dH}{dr}$. Also

$$V = \frac{C}{r^2}$$

Therefore,

$$\frac{dH}{dr} = -\rho \frac{C^2}{r^3}$$

This means that a change in r from r to $r+dr$ is accompanied by a loss in total energy per cu. ft. of the fluid of $\frac{\rho C^2}{r^3}$ which appears as an energy of vorticity in the region between r and $r+dr$ and this region acts as multiple roller bearings in stepping down the velocity.

CIRCULATION AROUND A CLOSED CURVE WHICH ENCLOSES ANY NUMBER OF VORTICES.

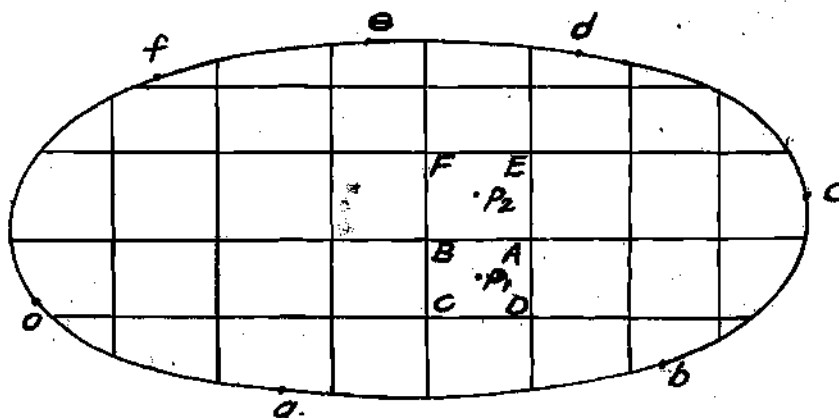


Fig. 18.

The line integral around the vortex whose center is at P is taken along the path ABCD and around the vortex ^{at} P_2 along the path AEFB. Now the directions of the velocities at the points are fixed and a line integral from A to B must have the opposite sign to one from B to A. But for the vortex

at P_1 , the integration was taken from A to B and for the vortex at P_2 from B to A. The circulation then about P_1 & P_2 is the line integral taken along the line AEFBC. It is apparent from this that in getting the algebraic sum of all the circulations around all the vortices within the closed curve abcdefo the line integral along any interior bound line will be taken twice, once in the positive direction and once in the negative direction, and the circulation along that line will be zero. Now the strength^g of a vortex being the circulation around it, the strength of all the vortices enclosed within the curve is the circulation around the curve oabcdefo because the exterior bound line is the only one along which the line integral was taken in one direction only. This is not a rigorous mathematical proof; it is put forward solely as an explanation.

THE VELOCITY POTENTIAL.

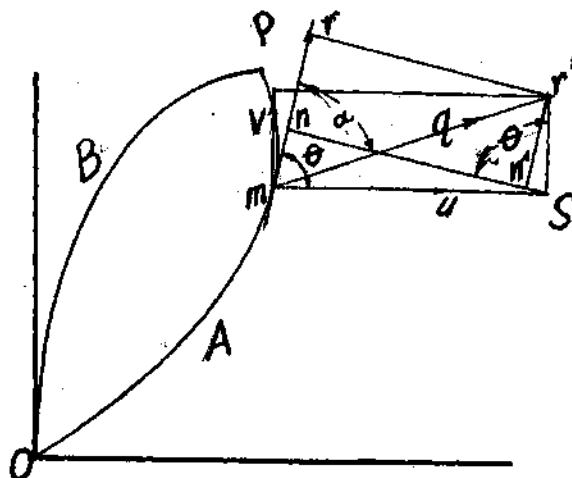


Fig. 19.

The circulation around the closed curve OAPBO has already been defined as the line integral of the tangential component of the velocity taken in the same sense as sequence of the letters OAPBO. That part of the line integral obtained from OAP we name ϕ_{OAP} and that part obtained from PBO, ϕ_{PBO} . If we took the line integral from o to P along OBP we would be integrating in a negative sense and therefore

$$\phi_{OBP} = - \phi_{PBO}$$

If there is no vorticity at any point inside the closed curve OAPBO the circulation around the curve is zero and therefore

$$\phi_{OAP} + \phi_{PBO} = 0$$

or

$$\phi_{OAP} - \phi_{OBP} = 0$$

which makes

$$\phi_{OAP} = \phi_{OBP}$$

Therefore, if there is no vorticity at any point in the plane there can be but one value of ϕ at any point in the plane and it makes no difference what curve the line integral is taken along so long as it joins o to P.

Fig. 19.

mr = tangential component of q to the curve OAP.

$$mn = u \cos \theta$$

$$sr' = v$$

$$nr = nr' = sr' \sin \theta = v \sin \theta$$

$$mr = mn + nr = u \cos \theta + v \sin \theta$$

$$= q \cos \alpha$$

so

$$q \cos \alpha = u \cos \theta + v \sin \theta$$

But

$$\phi_P = \int_{OAP} q \cos \alpha \, ds = \int_{OAP} (u \cos \theta \, ds + v \sin \theta \, ds)$$

Note:

$\int_{OAP} q \cos \alpha \, ds$ is the symbol for the line integral along the line OAP.

But

$$\begin{cases} \cos \theta = \frac{dx}{ds} \\ \sin \theta = \frac{dy}{ds} \end{cases}$$

and after making this substitution

$$\phi_P = \int_{OAP} u \frac{dx}{ds} \frac{ds}{1} + v \frac{dy}{ds} \frac{ds}{1} = \int_{OAP} (u dx + v dy)$$

The elementary part of ϕ , $d\phi$, at the point m (any point) is $u dx + v dy$.

$$d\phi = u dx + v dy$$

We already know that the components u and v depend upon the position of the point in the plane (they are functions of x and y). This makes ϕ a function of x and y . In mathematical language

$$\phi = F(x, y)$$

and

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy$$

Equating coefficients of like terms

$$\begin{cases} u = \frac{\partial \phi}{\partial x} \\ v = \frac{\partial \phi}{\partial y} \end{cases}$$

But

$$\begin{cases} u = \frac{\partial \psi}{\partial y} \\ v = -\frac{\partial \psi}{\partial x} \end{cases}$$

Therefore

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x} \end{aligned}$$

CONTINUITY.

If a fluid is incompressible and there is no source or sink within a region the quantity of fluid entering a region in a period of time must be exactly equal to the quantity which leaves during that same period. For such a region,

Outflow = Inflow.

Let the volume of the region be $dx \cdot dy \cdot 1$ ft.

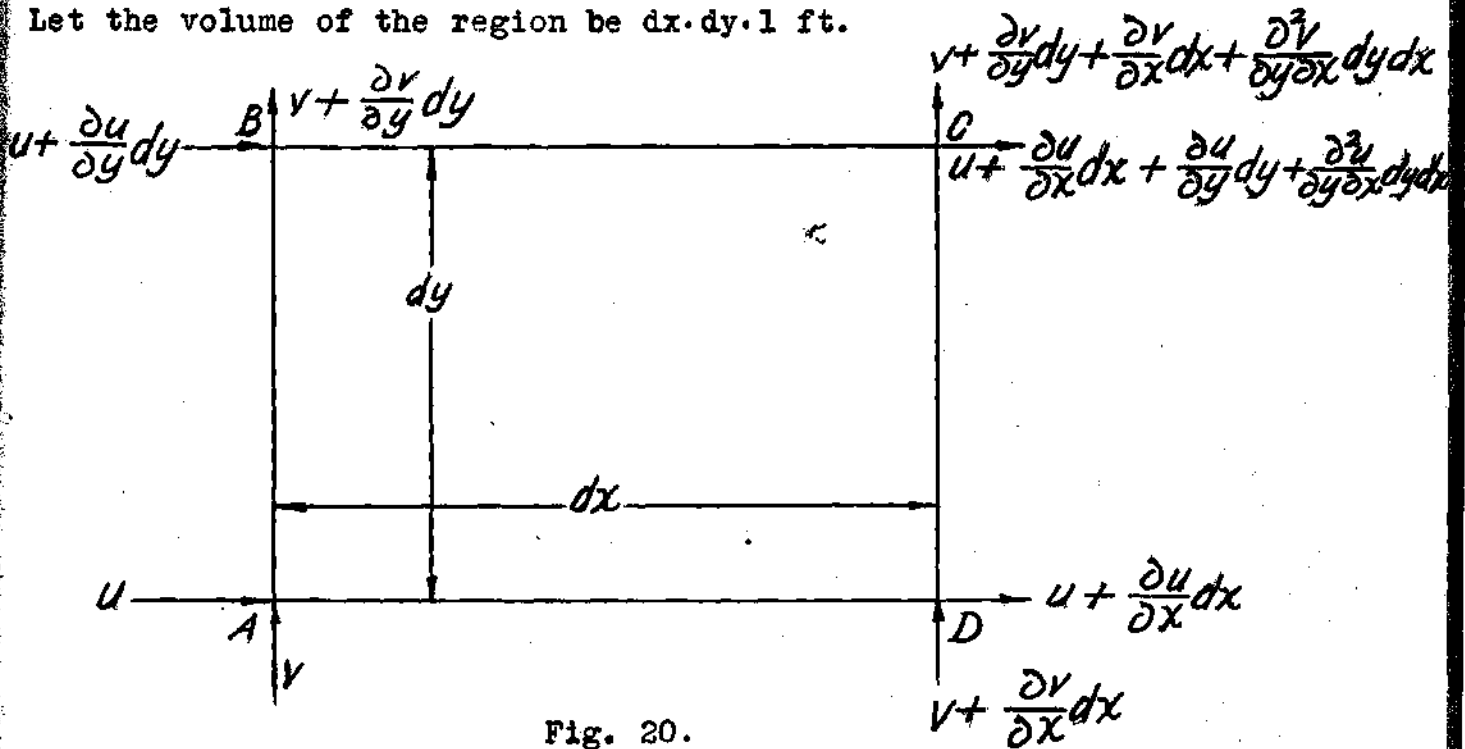


Fig. 20.

Inflow will be under AB and AD.

Outflow will be under BC and DC.

$$\text{Average velocity across the area AB} \cdot 1 \text{ ft.} = \frac{u + u + \frac{\partial u}{\partial y} dy}{2} = u + \frac{1}{2} \frac{\partial u}{\partial y} dy$$

$$\text{Average velocity across the area AD} \cdot 1 \text{ ft.} = v + \frac{1}{2} \frac{\partial v}{\partial x} dx$$

$$\text{Inflow} = u dy + \frac{1}{2} \frac{\partial u}{\partial y} dy^2 + v dx + \frac{1}{2} \frac{\partial v}{\partial x} dx^2$$

$$\text{Average velocity across the area BC} \cdot 1 \text{ ft.} = v + \frac{\partial v}{\partial y} dy + \frac{1}{2} \frac{\partial v}{\partial x} dx + \frac{1}{2} \frac{\partial^2 v}{\partial y \partial x} dy dx$$

$$\text{Average velocity across the area DC} \cdot 1 \text{ ft.} = u + \frac{\partial u}{\partial x} dx + \frac{1}{2} \frac{\partial u}{\partial y} dy + \frac{1}{2} \frac{\partial^2 u}{\partial y \partial x} dy dx$$

$$\text{Outflow} = v dx + \frac{\partial v}{\partial y} dy dx + \frac{1}{2} \frac{\partial v}{\partial x} dx^2 + \frac{1}{2} \frac{\partial^2 v}{\partial y \partial x} dy dx^2 + u dy + \frac{\partial u}{\partial x} dx dy + \frac{1}{2} \frac{\partial u}{\partial y} dy^2 + \frac{1}{2} \frac{\partial^2 u}{\partial y \partial x} dy dx^2$$

$$\text{Inflow} = -v dx - \frac{1}{2} \frac{\partial v}{\partial x} dx^2 - u dy - \frac{1}{2} \frac{\partial u}{\partial y} dy^2$$

$$0 = \frac{\partial v}{\partial y} dy dx + \frac{\partial u}{\partial x} dy dx$$

Since

$$\left. \begin{aligned} &\frac{1}{2} \frac{\partial^2 v}{\partial y \partial x} dy dx^2 \\ &\frac{1}{2} \frac{\partial^2 u}{\partial y \partial x} dy^2 dx \end{aligned} \right\} \text{are infinitesimals of higher order.}$$

The volume $dy \cdot dx \cdot 1$ ft. can not be zero.

Therefore

$$\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} = 0$$

Now

$$v = \frac{\partial \phi}{\partial y} \quad \text{so} \quad \frac{\partial v}{\partial y} = \frac{\partial^2 \phi}{\partial y^2}$$

and

$$u = \frac{\partial \phi}{\partial x} \quad \text{so} \quad \frac{\partial u}{\partial x} = \frac{\partial^2 \phi}{\partial x^2}$$

Therefore

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

And this is our criterion for continuity and is called Laplace's Equation for two dimensional motion.

CRITERION FOR UNIQUENESS OF VELOCITY POTENTIAL.

If the line integrals of the tangential components of velocity along all curves joining o to P are equal the velocity potential at P has one and only one value. It is therefore a unique value. In Fig. 19 it was shown that if there is no vorticity enclosed within the curve $OAPBO$ that

$$\phi_{OAP} = \phi_{OBP}$$

Hence, if there is no vorticity at any point in the plane ϕ will have a unique value at every point in the plane. This means that the criterion for circulation at any point in the plane must give a zero value, which means that there is no vorticity at any point.

$$\text{vorticity} = 2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$v = \frac{\partial \phi}{\partial y} \quad \text{so} \quad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right)$$

$$u = \frac{\partial \phi}{\partial x} \quad \text{so} \quad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right)$$

Therefore

$$\frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) - \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = 0$$

But this is the condition that

$$d\phi = udx + vdy$$

be a perfect or an exact differential.

In Differential Equations an equation is said to be exact if it is a perfect differential. Thus $dF = Mdx + Ndy$ is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

In our case

$$\begin{aligned} M = u = \frac{\partial \phi}{\partial x} \quad \text{so} \quad \frac{\partial M}{\partial y} &= \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) \\ N = v = \frac{\partial \phi}{\partial y} \quad \text{so} \quad \frac{\partial N}{\partial x} &= \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) \end{aligned}$$

Note: ϕ is assumed to be a continuous function with continuous partial derivatives.

$$\therefore \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) \equiv \frac{\partial^2 \phi}{\partial y \partial x}$$

ILLUSTRATION OF AN EXACT DIFFERENTIAL.

Let $F = x^3 y^2$

$$\begin{aligned} dF &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = Mdx + Ndy \\ &= 3x^2 y^2 dx + 2x^3 y dy \end{aligned}$$

$$M = 3x^2 y^2 \quad \text{so} \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x} \right) = 6x^2 y$$

$$N = 2x^3 y \quad \text{so} \quad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial y} \right) = 6x^2 y$$

From all that has been illustrated and proven a flow will have a unique value of velocity potential at every point in the plane if the right hand side of the equation for $d\phi$ is an exact differential.

EQUATIONS OF VELOCITY POTENTIAL FOR A FEW USEFUL TYPES OF FLOW.

1. Uniform flow parallel to the axis of x:

$$\psi = Uy$$

$$\begin{cases} \frac{\partial \psi}{\partial y} = U = \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial x} = 0 = \frac{\partial \phi}{\partial y} \end{cases}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = U dx + 0 \cdot dy$$

$$\phi = Ux + \text{constant}$$

Assign any value to constant, usually taken to be zero.

2. Source at origin:

$$\psi = \frac{m}{2\pi} \theta$$

$$\begin{cases} \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r} \frac{m}{2\pi} = u' = \frac{\partial \phi}{\partial r} \\ -\frac{\partial \psi}{\partial r} = 0 = v' = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \end{cases}$$

$$d\phi = u' dr + v' r d\theta$$

$$= \frac{m}{2\pi} \cdot \frac{dr}{r} + 0 d\theta$$

$$\phi = \frac{m}{2\pi} \log r$$

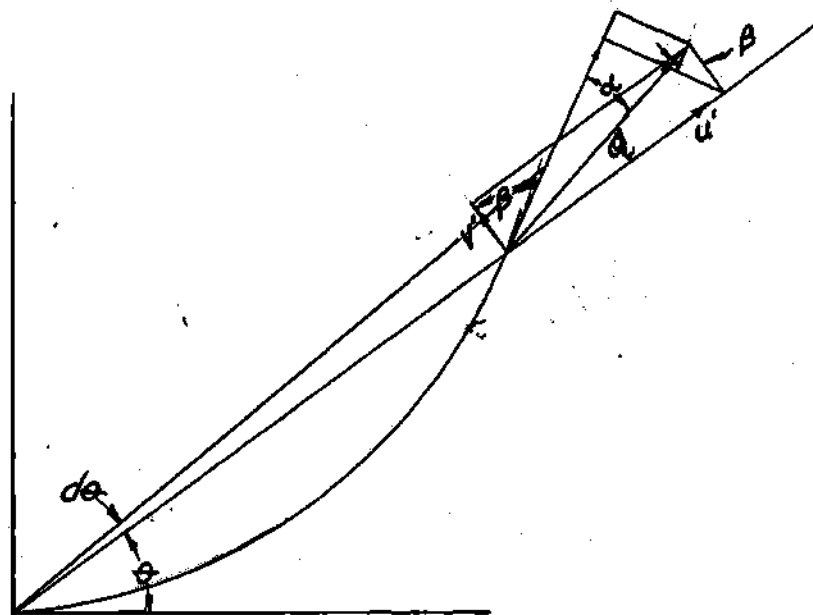
3. Doublet at the origin with axis along the axis of x:

$$\psi = \frac{\mu}{2\pi} \cdot \frac{y}{x^2 + y^2}$$

or

$$\psi = \frac{\mu}{2\pi} \cdot \frac{\sin \theta}{r}$$

$$\begin{cases} \frac{\partial \psi}{\partial \theta} = u' r = \frac{\mu}{2\pi} \frac{\cos \theta}{r} \\ \frac{\partial \psi}{\partial r} = -v' = -\frac{\mu}{2\pi} \frac{\sin \theta}{r^2} \end{cases}$$



$$q \cos \alpha \, ds = (u' \sin \beta + v' \cos \beta) \, ds$$

$$\text{But } \begin{cases} \sin \beta = \frac{dr}{ds} \\ \cos \beta = \frac{r d\theta}{ds} \end{cases}$$

substituting the latter relation

$$\begin{aligned} q \cos \alpha \, ds &= u' \frac{dr}{ds} ds + v' \frac{r d\theta}{ds} ds \\ &= d\phi = u' dr + v' r d\theta = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta \end{aligned}$$

Equating coefficients of like independent variables

$$\begin{cases} u' = \frac{\partial \phi}{\partial r} \\ v' r = \frac{\partial \phi}{\partial \theta} \end{cases}$$

$$\text{substituting } \begin{cases} \frac{\mu}{2\pi} \frac{\cos \theta}{r^2} \text{ for } u' \\ \frac{\mu}{2\pi} \frac{\sin \theta}{r^2} \text{ for } v' \end{cases}$$

$$d\phi = \frac{\mu}{2\pi} \frac{\cos \theta}{r^2} dr + \frac{\mu}{2\pi} \frac{\sin \theta}{r} d\theta$$

This is an exact differential equation since

$$\frac{\partial}{\partial \theta} \left(\frac{\mu}{2\pi} \frac{\cos \theta}{r^2} \right) = -\frac{\mu}{2\pi} \frac{\sin \theta}{r^2} = \frac{\partial}{\partial r} \left(\frac{\mu}{2\pi} \frac{\sin \theta}{r} \right) = -\frac{\mu}{2\pi} \frac{\sin \theta}{r^2}$$

Solution: Integrate $\frac{\partial \phi}{\partial r} dr$, keeping θ constant, and introduce a function of θ , $[F(\theta)]$, instead of a constant, integration.

Differentiate the result with respect to θ and equate to $\frac{\partial \phi}{\partial \theta}$

. This latter will determine $F(\theta)$.

$$\int \frac{\partial \phi}{\partial r} dr = \int \frac{\mu}{2\pi} \frac{\cos \theta}{r^2} dr = -\frac{\mu}{2\pi} \frac{\cos \theta}{r} + F(\theta)$$

$$\frac{\partial F(\theta)}{\partial \theta} + \frac{\mu}{2\pi} \frac{\sin \theta}{r} = \frac{\mu}{2\pi} \frac{\sin \theta}{r}$$

$$\text{and } \frac{\partial F(\theta)}{\partial \theta} d\theta = 0$$

and

$$F(\theta) = \text{constant.}$$

substituting,

$$\phi = -\frac{\mu}{2\pi} \frac{\cos \theta}{r} + \text{constant.}$$

Let $\phi = 0$ where $\begin{cases} \theta = 0 \\ r = \infty \end{cases}$ so that constant = 0

Therefore,

$$\phi = -\frac{\mu}{2\pi} \frac{\cos \theta}{r}$$

4. For the flow

$$\psi = -U \sin \theta \left(r - \frac{a^2}{r} \right) - \frac{K}{2\pi} \log \frac{r}{a}$$

$$\begin{cases} u'r = \frac{\partial \psi}{\partial \theta} = -U \cos \theta \left(r - \frac{a^2}{r} \right) = r \frac{\partial \phi}{\partial r} \\ v' = -\frac{\partial \psi}{\partial r} = U \sin \theta \left(1 + \frac{a^2}{r^2} \right) + \frac{K}{2\pi} \cdot \frac{1}{r} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} \end{cases}$$

$$d\phi = \frac{\partial \phi}{\partial r} dr + \frac{\partial \phi}{\partial \theta} d\theta$$

$$\int \frac{\partial \phi}{\partial r} dr = \int -U \cos \theta \left(1 - \frac{a^2}{r^2}\right) dr = -U \cos \theta \left(r + \frac{a^2}{r}\right) + F(\theta)$$

$$\frac{\partial F(\theta)}{\partial \theta} + U \sin \theta \left(r + \frac{a^2}{r}\right) = \frac{\partial \phi}{\partial \theta} = U \sin \theta \left(r + \frac{a^2}{r}\right) + \frac{K}{2\pi}$$

$$\int \frac{\partial F(\theta)}{\partial \theta} d\theta = \int \frac{K}{2\pi} d\theta$$

$$F(\theta) = \frac{K}{2\pi} \theta + \text{constant}$$

substituting,

$$\phi = -U \cos \theta \left(r + \frac{a^2}{r}\right) + \frac{K}{2\pi} \theta + \text{constant.}$$

Let $\phi = 0$ where $\frac{K\theta}{2\pi} = U \cos \theta \left(r + \frac{a^2}{r}\right)$ then the constant = 0.

Therefore,

$$\phi = -U \cos \theta \left(r + \frac{a^2}{r}\right) + \frac{K}{2\pi} \theta$$

5. For the assumed flow

$$\psi = \frac{C}{r^2} = \frac{C}{x^2 + y^2}$$

$$\begin{cases} \frac{\partial \psi}{\partial y} = u = \frac{-2Cy}{(x^2 + y^2)^2} = \frac{\partial \phi}{\partial x} \\ -\frac{\partial \psi}{\partial x} = v = \frac{2Cx}{(x^2 + y^2)^2} = \frac{\partial \phi}{\partial y} \end{cases}$$

$$\text{But } \begin{cases} \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = \frac{-2C(x^2 + y^2) + 8Cy^2}{(x^2 + y^2)^3} = \frac{6Cy^2 - 2Cx^2}{(x^2 + y^2)^3} \\ \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right) = \frac{+2C(x^2 + y^2) - 8Cx^2}{(x^2 + y^2)^3} = \frac{2Cy^2 - 6Cx^2}{(x^2 + y^2)^3} \end{cases}$$

In this case

$$\frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) \text{ is not equal to } \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right)$$

and there is no unique value of ϕ for every point in the plane.

There will be as many values of ϕ at a point as there are different curves joining the origin to the point.

LINES OF CONSTANT VELOCITY POTENTIAL.

For any fluid flow there are lines representing constant values of ϕ just as there are lines which represent constant values of ψ . Furthermore, the lines of constant values of ϕ intersect those of constant values of ψ except at stagnation points, (points where the resultant velocity is zero), always at ninety degrees.

$$\psi = \psi(x, y)$$

$$\phi = \phi(x, y)$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -v dx + u dy$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy$$

On a line $\psi = \psi_1 = \text{constant}$, $d\psi = 0$

For any such line

$$0 = -v dx + u dy$$

$$\frac{dy}{dx} = \frac{v}{u}$$

On a line $\phi = \phi_1 = \text{constant}$, $d\phi = 0$

For any such line

$$0 = u dx + v dy$$

$$\frac{dy}{dx} = -\frac{u}{v}$$

The line of ϕ_1 will intersect the line of ψ_1 at some point (x_1, y_1) . These co-ordinates when substituted in the equations of u and v will give u_1 and v_1 . The slope of ψ_1 at the point of intersection will be

$$\left(\frac{dy}{dx} \right) = \frac{v_1}{u_1}$$

where $\psi = \psi_1$

$$x = x_1$$

$$y = y_1$$

The slope of ϕ_1 at the point of intersection will be

$$\left(\frac{dy}{dx} \right) = -\frac{u_1}{v_1}$$

$\phi = \phi_1$
 $x = x_1$
 $y = y_1$

The two slopes at the point of intersection are negative reciprocals and the tangents are therefore perpendicular to each other.

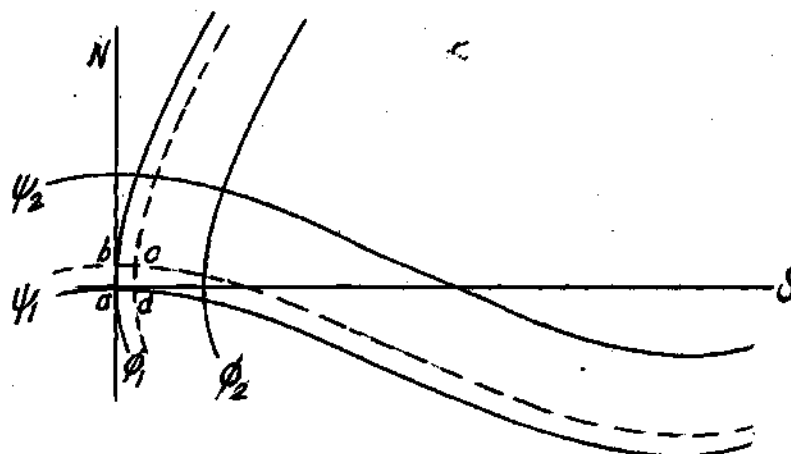


Fig. 21.

At a point on a stream line:

let distances parallel to the tangent be represented by s .

let distances perpendicular to the tangent be represented by n .

let ψ for the stream line shown as a dotted line be $\psi_1 + d\psi$.

let ϕ for the potential line shown as a dotted line be $\phi_1 + d\phi$.

There can be no component of the resultant velocity q parallel to n , and consequently

$$d\psi = \frac{d\psi}{dn} dn + 0 \cdot ds$$

But $d\psi$ quantity of fluid passing through the area $dn \cdot 1$ ft. per unit time.

and $\frac{Q}{A} = V$

which makes

$$\frac{d\psi}{dn} = q$$

similarly

$$\frac{d\phi}{ds} = q = \frac{d\psi}{dn}$$

If $d\phi$ be taken equal to $d\psi$, ds will be equal to dn which will cause the streamlines and potential lines to form a network of small squares. Because at any point ds must be taken parallel to the stream line at that point and dn perpendicular to the stream line at that point the sides of visible and measurable four sided areas will be curved except in a few special cases. The angles, however, will always be right angles. These facts regarding elementary squares formed by intersecting stream and potential lines will be very helpful in understanding the relation between the flow around a cylinder and the flow around an airfoil, the cylinder having been transformed into an airfoil and the flow having been transformed from one around the cylinder to another around the airfoil.

THE COMPLEX VARIABLE.

In the solution of the quadratic equation

$$z^2 - 2az + c = 0$$

$$z_1 = [a + \sqrt{a^2 - c}] \quad \text{and} \quad z_2 = [a - \sqrt{a^2 - c}]$$

$$\text{If } \begin{cases} a^2 = c \\ \text{or} \\ a^2 - c = \pm b^2 \end{cases} \quad \text{then the corresponding values of}$$

$$z_1 \text{ are } \begin{cases} z = a \\ \text{and} \\ z = a \pm b \end{cases}$$

but if $a^2 - c = -b^2$, the value of z is $z = a + b\sqrt{-1}$. For centuries, mathematicians called this value of z an imaginary root but Gauss, Argand, and Wessel working independently gave a meaning to this expression by perceiving that it could represent a point in a plane. The " a " distance they called real and measured it along the x axis but the " b " distance they called imaginary and measured it along the y axis and denoted it ib . The x axis is, therefore, called the axis of reals and the y axis, the axis of imaginaries.

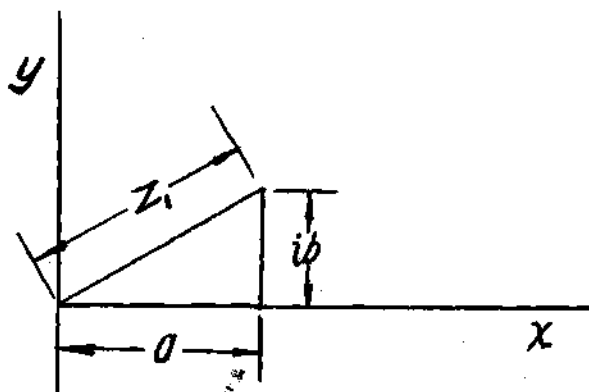


Fig. 22.

The absolute value of z is indicated by $|z|$ and means that only a distance is considered, the sign being suppressed.

$$|z| = \sqrt{a^2 + b^2}$$

The fact that such equations as the one cited always give two imaginary roots which are conjugate allows a simple method of determining $|z|$.

$$\text{Let } z = x + iy$$

$$\text{and } \bar{z} = x - iy$$

$$\text{then } |z|^2 = z \cdot \bar{z} = x^2 - i^2 y^2$$

$$\text{but } i^2 = [(-1)^{\frac{1}{2}}]^2 = -1$$

$$\therefore |z| = \sqrt{x^2 + y^2}$$

This method has elegance and power in many mathematical analyses.

The following analysis carried out by Euler gives another very useful, if not the most useful, expression for z .

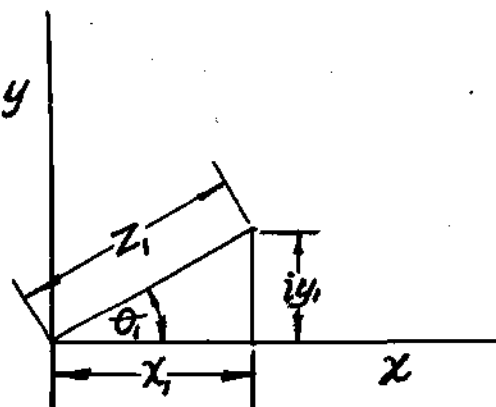


Fig. 23.

$$\text{Let } |z| = r,$$

$$x = r \cos \theta,$$

$$y = r \sin \theta,$$

so that in general

$$z = r(\cos \theta + i \sin \theta)$$

z depends on both r and θ and an infinitesimal increment in z will be

$$\begin{aligned} dz &= \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta \\ &= (\cos \theta + i \sin \theta) dr + r(-\sin \theta + i \cos \theta) d\theta \end{aligned}$$

but $-1 = i^2$ and we may write

$$dz = (\cos \theta + i \sin \theta) dr + r(i^2 \sin \theta + i \cos \theta) d\theta$$

$$\begin{aligned}
 &= (\cos \theta + i \sin \theta) dr + r(\cos \theta + i \sin \theta) i d\theta \\
 &= r(\cos \theta + i \sin \theta) \frac{dr}{r} + r(\cos \theta + i \sin \theta) i d\theta
 \end{aligned}$$

rewriting

$$\frac{dz}{r(\cos \theta + i \sin \theta)} = \frac{dz}{z} = \frac{dr}{r} + i d\theta.$$

But this last equation is a differential equation every term of which is integrable directly.

Integrating

$$\log_e z = \log_e r + i\theta$$

Because $\log_e e = 1$

$$\begin{aligned}
 \log_e z &= \log_e r + i\theta \log_e e = \log_e r + \log_e e^{i\theta} \\
 &= \log_e re^{i\theta}
 \end{aligned}$$

or

$$z = re^{i\theta}.$$

The constant of integration can be made zero.

We now have

$$z = re^{i\theta} = r(\cos \theta + i \sin \theta)$$

so that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

ADDITION OF COMPLEX NUMBERS.

Let $z_1 = a + ib$

and $z_2 = c + id$

$$z_1 + z_2 = (a + c) + i(b + d)$$

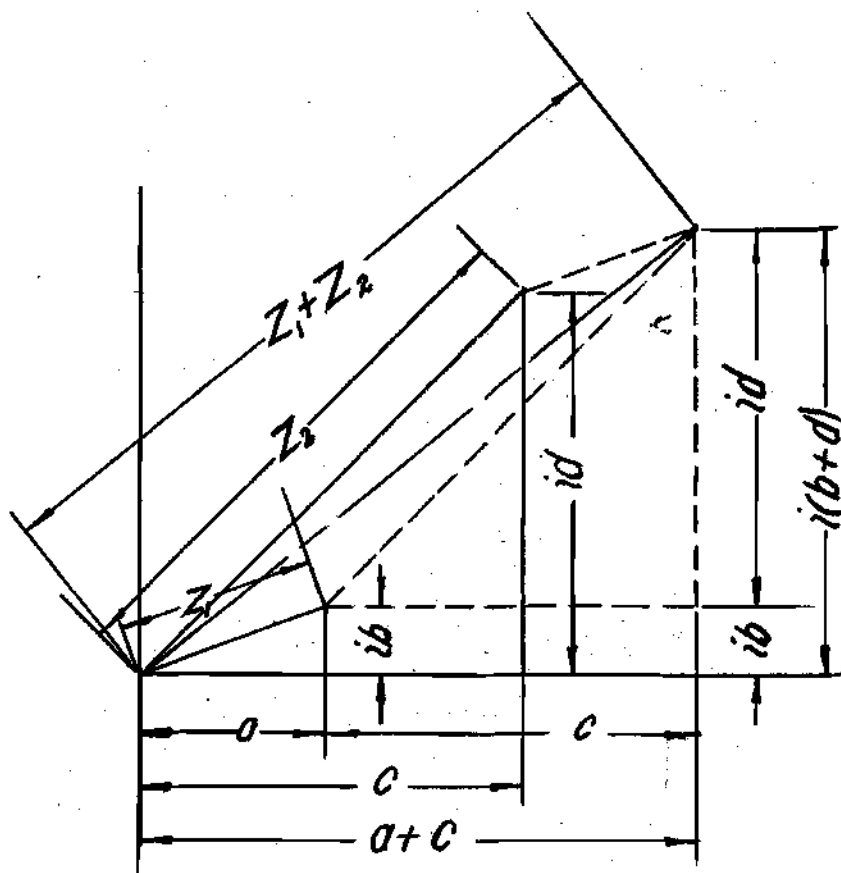


Fig. 24.

SUBTRACTION OF COMPLEX NUMBERS.

Let $z_1 = a + ib$

and $z_2 = c + id$

$$z_1 - z_2 = (a - c) + i(b - d)$$

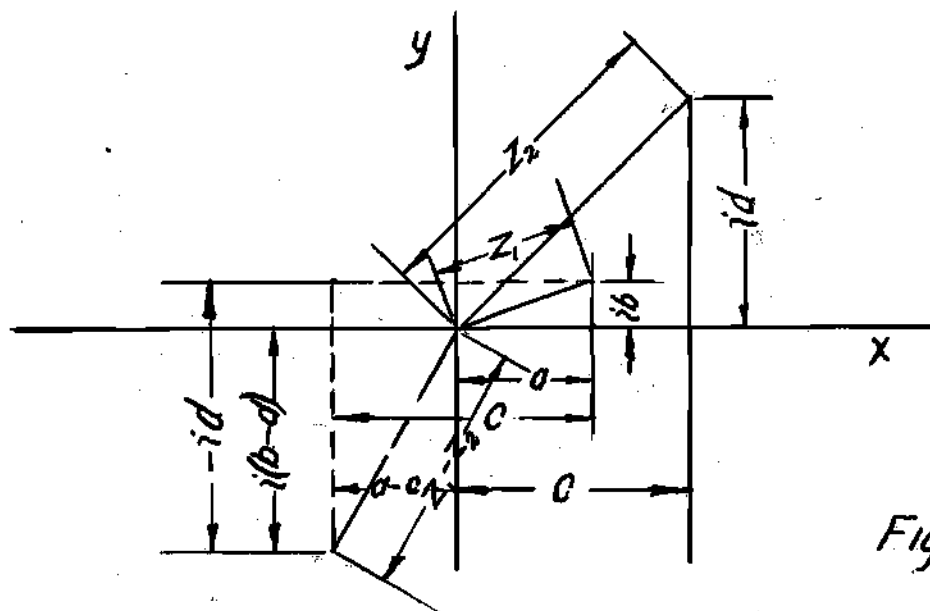


Fig. 25.

MULTIPLICATION OF COMPLEX NUMBERS.

The form of a complex number arrived at by Euler is the best one for multiplication, division, raising to a power, extraction of a root, etc.

$$\begin{aligned} \text{Let } z_1 &= r_1 e^{i\theta_1} \\ \text{and } z_2 &= r_2 e^{i\theta_2} \\ \hline z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

The absolute value of the product is equal to the product of the absolute values and the angle of the product is equal to the sum of the angles of the complex numbers.

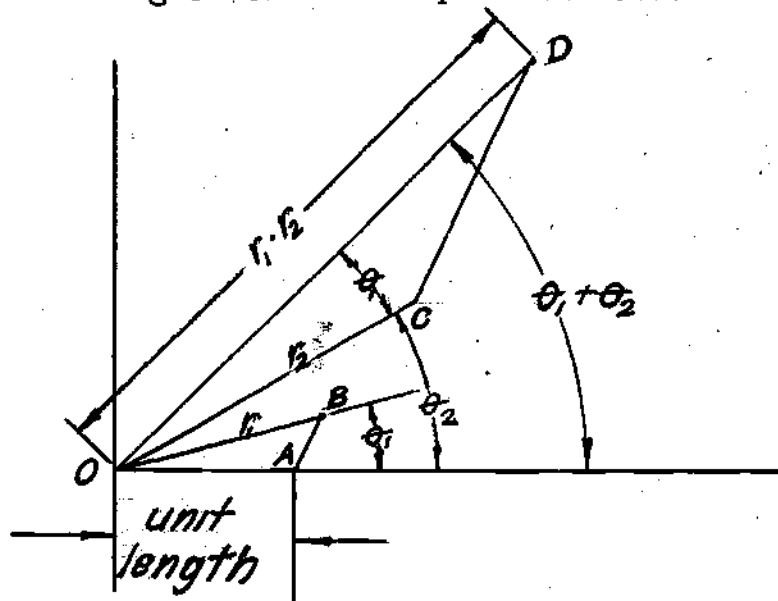


Fig. 26.

$\triangle OCD$ constructed similar to $\triangle OAB$. We can write because of similarity

$$\frac{OB}{OA} = \frac{OD}{OC} \quad \text{or} \quad \frac{OB}{1} = \frac{OD}{OC}.$$

making

$$OD = OB \cdot OC$$

but $OD = z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$

DIVISION OF COMPLEX NUMBERS.

$$\text{Let } z_1 = r_1 e^{i\theta_1}$$

$$\text{and } z_2 = r_2 e^{i\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

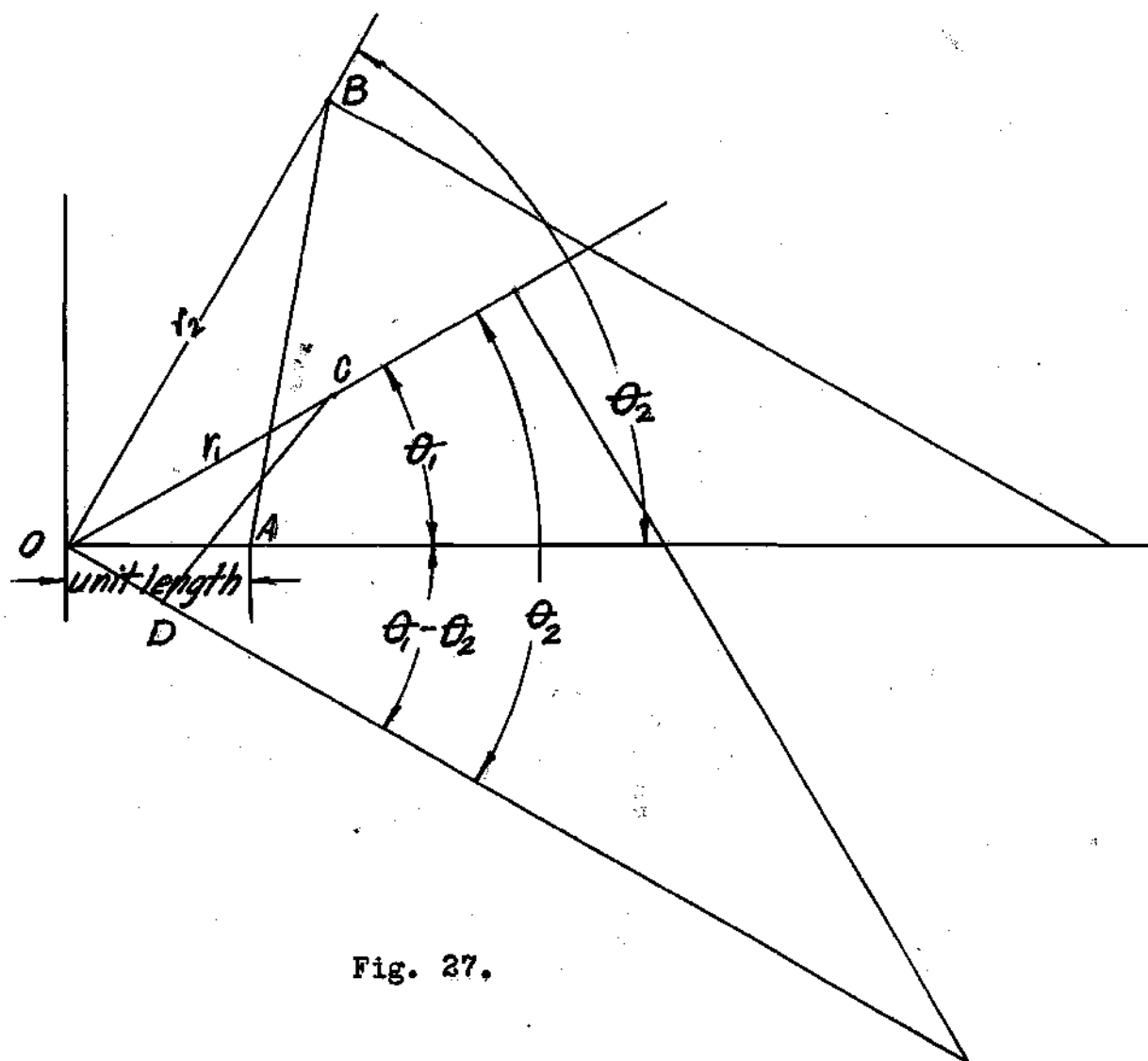


Fig. 27.

$\triangle OCD$ constructed similar to $\triangle OAB$.

$$\angle OCD = \angle OBA$$

$$\angle DOC = \angle AOB$$

$$\frac{OD}{OA} = \frac{OC}{OB} = \frac{OD}{1}$$

$$OD = \frac{OC}{OB} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

The absolute value of the quotient is equal to the absolute value of the dividend divided by the absolute value of the divisor and the angle of the quotient is equal to the angle of the dividend minus the angle of the divisor.

RAISING THE POWER OF A COMPLEX NUMBER.

$$z = re^{i\theta}$$

$$z^2 = z \cdot z = re^{i\theta} \cdot re^{i\theta} = r^2 e^{i2\theta}$$

We could have taken the product of the more primary forms and gotten the same result.

$$\begin{aligned} z^2 &= (r \cos \theta + ir \sin \theta)^2 \\ &= r^2 \cos^2 \theta + i2r^2 \sin \theta \cos \theta + i^2 r^2 \sin^2 \theta \end{aligned}$$

since

$$i^2 = -1$$

$$z^2 = r^2 \left[(\cos^2 \theta - \sin^2 \theta) + i(\sin \theta \cos \theta + \cos \theta \sin \theta) \right]$$

$$\begin{cases} \cos^2 \theta - \sin^2 \theta = \cos 2\theta \\ \sin \theta \cos \theta + \cos \theta \sin \theta = \sin 2\theta \end{cases}$$

$$z^2 = r^2(\cos 2\theta + i \sin 2\theta) = r^2 e^{i2\theta}$$

$$z^3 = r^3 e^{i3\theta} = r^3(\cos 3\theta + i \sin 3\theta)$$

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

EXTRACTING THE ROOT OF A COMPLEX NUMBER.

If $(\rho e^{i\phi})^n = r e^{i\theta}$

Then $\rho e^{i\phi}$ is one of the n roots of $r e^{i\theta}$.

$$\rho^n e^{in\phi} = r e^{i\theta}$$

If these two are equal

$$\rho^n = r$$

or

$$\rho = r^{\frac{1}{n}}$$

but

$$n\phi = \theta; \theta \pm 2\pi; \theta \pm 4\pi; \theta \pm 6\pi; \dots \theta \pm 2(n-1)\pi$$

or

$$\phi = \frac{\theta}{n}; \frac{\theta \pm 2\pi}{n}; \frac{\theta \pm 4\pi}{n}; \dots \frac{\theta \pm 2(n-1)\pi}{n}$$

The necessity for n values of ϕ can be grasped easily by use of an illustration.

Let $z = 8e^{i\frac{\pi}{6}}$

$$z^{\frac{1}{3}} = 2e^{i\frac{\pi}{18}}; 2e^{i(\frac{\pi}{18} + \frac{2\pi}{3})}; 2e^{i(\frac{\pi}{18} + \frac{4\pi}{3})}$$

$$= 2e^{i\frac{\pi}{18}}; 2e^{i(\frac{13\pi}{18})}; 2e^{i(\frac{25\pi}{18})}$$

$$(2e^{i\frac{\pi}{18}})^3 = 8e^{i\frac{\pi}{6}}$$

$$(2e^{i\frac{13\pi}{18}})^3 = 8e^{i\frac{13\pi}{6}} = 8e^{i[2\pi + \frac{\pi}{6}]} = 8e^{i\frac{\pi}{6}}$$

$$(2e^{i\frac{25\pi}{18}})^3 = 8e^{i\frac{25\pi}{6}} = 8e^{i[4\pi + \frac{\pi}{6}]} = 8e^{i\frac{\pi}{6}}$$

Graph of the three roots

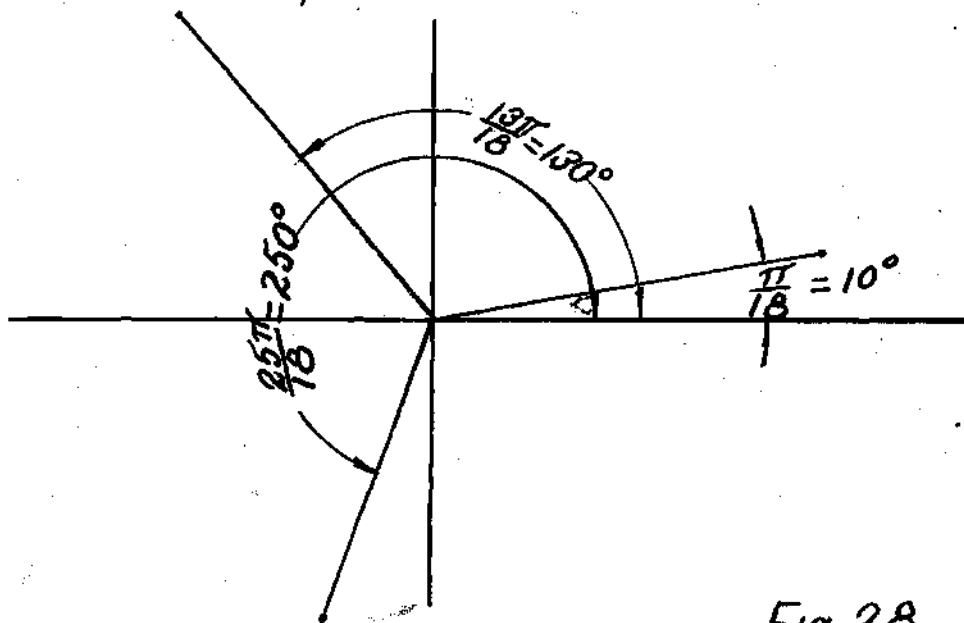


Fig. 28.

The angle between any two adjacent roots is 120° .

An interesting application of this is the extraction of any root of $[-1]$.

$$-1 = 1e^{i\pi} = \cos\pi + i\sin\pi = -1 + i \cdot 0$$

$$(e^{i\pi})^{\frac{1}{2}} = e^{i\frac{\pi}{2}}; e^{i(\frac{\pi}{2} + 2\pi)}$$

$$= e^{i\frac{\pi}{2}}; e^{i\frac{5}{2}\pi}$$

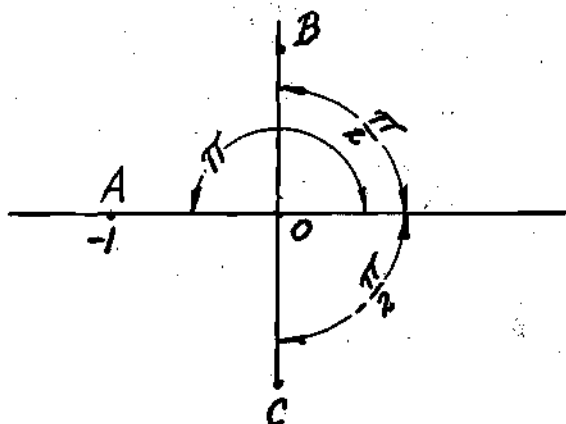


Fig. 29.

$$OA = 1 \cdot e^{i\pi}$$

$$OB = 1 \cdot e^{i\frac{\pi}{2}} = \sqrt{-1}$$

$$OC = 1 \cdot e^{i\frac{3\pi}{2}} = -\sqrt{-1}$$

EXPONENTIALS AND LOGARITHMS OF COMPLEX NUMBERS.

$$e^{i\phi} = \cos \phi + i \sin \phi$$

$$e^{-i\phi} = \cos(-\phi) + i \sin(-\phi) = \cos \phi - i \sin \phi$$

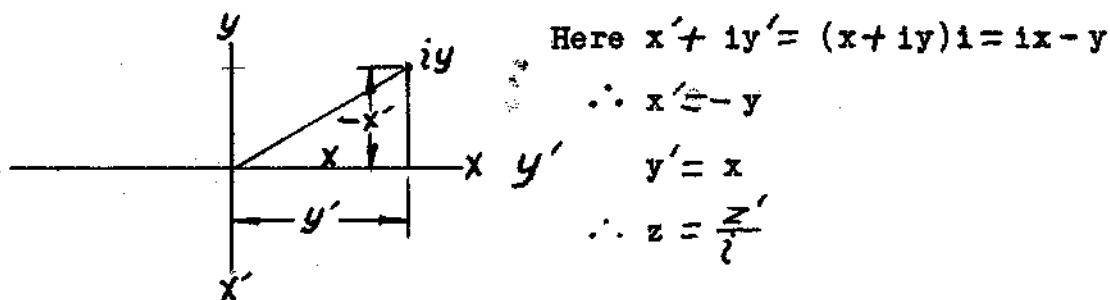
Adding: $e^{i\phi} + e^{-i\phi} = 2 \cos \phi$ or $\cos \phi = \frac{e^{i\phi} + e^{-i\phi}}{2}$

Subtracting: $e^{i\phi} - e^{-i\phi} = 2i \sin \phi$ or $\sin \phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}$

This theory can be extended to include $\phi = z$

$$\text{so that } \begin{cases} \cos z = \frac{e^{iz} + e^{-iz}}{2} \\ \sin z = \frac{e^{iz} - e^{-iz}}{2i} \end{cases} \quad (a.)$$

Next substitute $z' = iz$ which is equivalent to rotating the x and y axes backward through 90° .



We get $\frac{e^{z'} + e^{-z'}}{2} = \cos \frac{z'}{i} = \cos(-iz') = \cos(iz') = \cosh z'$
 $\frac{e^{z'} - e^{-z'}}{2i} = \sin(-iz') = -\frac{\sinh z'}{i}$

$$\therefore \cos(iz') = \cosh z'$$

$$i \sin(-iz') = \sinh z' \text{ or } \sin(iz') = i \sinh z'$$

$$\therefore \cos(iz) = \cosh z$$

$$\sin(iz) = i \sinh z$$

(b.)

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$\cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

From (b) on substituting $z' = iz$ we obtain

$$\cos z' = \cosh iz'$$

$$\sin z' = i \sinh \frac{z'}{i} = i \sinh (-iz') = -i \sinh iz'$$

$$\therefore \cosh iz = \cos z$$

$$\sinh iz = i \sin z$$

Note that $-\sinh(iz) = \sinh(-iz)$

$$\text{for } \sinh(-iz) = \frac{e^{-iz} - e^{-(-iz)}}{2} = -\frac{e^{iz} - e^{-iz}}{2}$$

$$\text{Also } \cosh(-iz) = \frac{e^{-iz} + e^{-(-iz)}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cosh iz$$

Squaring both sides of both Equations in (b)

$$\cosh^2 z = \cos^2(iz)$$

$$-\sinh^2 z = \sin^2(iz)$$

$$\therefore \cosh^2 z - \sinh^2 z = 1$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

Put

$$z'_1 = iz_1$$

$$z'_2 = iz_2$$

$$\sin[-i(z'_1 + z'_2)] = \sin(-iz'_1) \cos(-iz'_2) + \cos(-iz'_1) \sin(-iz'_2)$$

$$-i \sinh(z'_1 + z'_2) = -i \sinh z'_1 \cosh z'_2 - i \cosh z'_1 \sinh z'_2$$

$$\therefore \sinh(z'_1 + z'_2) = \sinh z'_1 \cosh z'_2 + \cosh z'_1 \sinh z'_2$$

In like manner:

$$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$$

$$\text{By letting } \begin{cases} z_1 = x \\ z_2 = iy \end{cases}$$

$$\sinh(x + iy) = \sinh x \cosh(iy) + \cosh x \sinh iy$$

$$= \cos y \sinh x + i \sin y \cosh x$$

$$\cosh(x + iy) = \cos y \cosh x + i \sin y \sinh x$$

$$\text{By letting } \begin{cases} z_1 = z \\ z_2 = 2k\pi i \end{cases}$$

$$\sinh(z + 2k\pi i) = \sinh z \cosh 2k\pi i + \cosh z \sinh 2k\pi i$$

$$= \cos 2k\pi \sinh z + [\sin 2k\pi] \cosh z = \sinh z$$

$$\cosh(z + 2k\pi i) = \cosh z$$

LOGARITHMS:

$$\text{If } e^w = z$$

then we define w as $\log z$

$$1. \log(z_1 \cdot z_2) = \log z_1 + \log z_2$$

$$2. \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

$$3. \log z^n = n \log z, \text{ etc.}$$

Where k is an integer

$$e^{w+2k\pi i} = e^w \cdot e^{2k\pi i} = e^w (\cos 2k\pi + i \sin 2k\pi) = e^w = z$$

The logarithm of a complex number is therefore infinitely mul-

multiple valued.

$$e^w = re^{i\phi}$$

$$w = \log(re^{i\phi}) = \log r + \log e^{i(\phi + 2k\pi)} = \log r +$$

$$i(\phi + 2k\pi) \log e = \log r + i(\phi + 2k\pi)$$

By this definition a negative number now has a logarithm.

$$\log(-a) = \log(-1) + \log a = \log a + \log e^{(2k-1)\pi i} =$$

$$\log a + (2k-1)\pi i \log e$$

$$\text{since } e^{(2k-1)\pi i} = \cos(2k-1)\pi + i \sin(2k-1)\pi = -1 + i0$$

$$\log(-a) = \log a + (2k-1)\pi i$$

INVERSE HYPERBOLIC FUNCTIONS:

$$\text{If } \begin{cases} z = \sinh w \\ z = \cosh w \\ z = \tanh w \end{cases} \quad \text{then } \begin{cases} w = \sinh^{-1} z \\ w = \cosh^{-1} z \\ w = \tanh^{-1} z \end{cases}$$

Definition

$$\sinh w = \frac{e^w - e^{-w}}{2} = z$$

Rearranging

$$e^{2w} - 2ze^w - 1 = 0$$

Solving

$$e^w = z \pm \sqrt{z^2 + 1}$$

$$\therefore w = \log(z \pm \sqrt{1 + z^2}) = \sinh^{-1} z$$

If z is real and equals x , $w = \log(x + \sqrt{1 + x^2})$

Definition

$$\cosh w = \frac{e^w + e^{-w}}{2} = z$$

$$e^{2w} - 2e^w z + 1 = 0$$

$$e^w = z \pm \sqrt{z^2 - 1}$$

$$\therefore w = \log(z \pm \sqrt{z^2 - 1}) = \cosh^{-1} z$$

Definition

$$\tanh w = \frac{\sinh w}{\cosh w} = \frac{e^{2w} - 1}{e^{2w} + 1} = z$$

$$z(e^{2w} + 1) = e^{2w} - 1$$

$$e^{2w}(z - 1) = -1 - z$$

$$e^{2w} = \frac{1+z}{1-z} ; \therefore e^w = \sqrt{\frac{1+z}{1-z}} \text{ and } w = \frac{1}{2} \log \frac{1+z}{1-z} = \tanh^{-1} z$$

THE DERIVATIVE OF A COMPLEX FUNCTION.

Let $\zeta = \xi + i\eta$

and $z = x + iy$

ζ plane

z plane

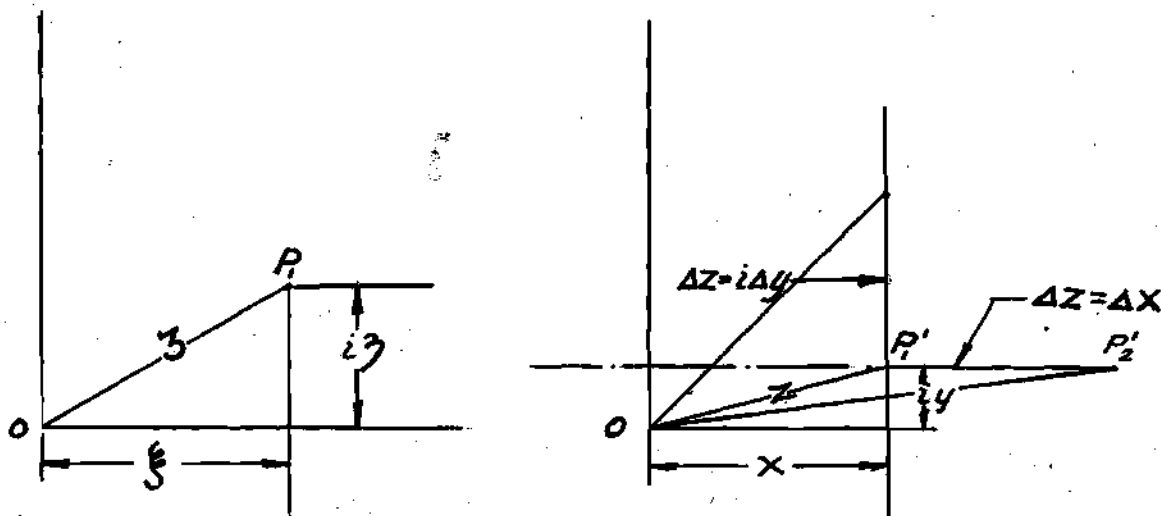


Fig. 30

Suppose that there is such a relation between the complex number ζ in the ζ plane and the complex number z in the z plane that when the point P_1' is chosen in the z plane the

point P, will be determined in the z plane. This would mean that

$$z = f(z)$$

$$\Delta z = \Delta x + i\Delta y$$

$$\Delta z = \Delta x + i\Delta y$$

$$\frac{\Delta f}{\Delta z} = \frac{\Delta f + i\Delta g}{\Delta x + i\Delta y}$$

Suppose that z takes on increments in such a manner that $\Delta z = \Delta x$. This is possible for z could have an infinite number of values drawn from the origin to any point on the line through P' parallel to the x axis. For this method of varying z , $\Delta y = 0$.

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f + i\Delta g}{\Delta x + i\cdot 0} = \frac{\partial f}{\partial x} + i \frac{\partial g}{\partial x}$$

In a similar manner let z take on increments in such a manner that $z = i\Delta y$. Therefore $\Delta x = 0$

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta y \rightarrow 0} \frac{\Delta f + i\Delta g}{0 + i\Delta y} = \frac{1}{i} \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$$

If the $\lim_{\Delta z \rightarrow 0} \frac{\Delta f}{\Delta z}$ has any meaning both the preceding limits must approach the same value for the point P' . It is a necessary condition, therefore, if the limit is to have a unique value that

$$\frac{\partial f}{\partial x} + i \frac{\partial g}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} = \frac{i}{i^2} \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} = -i \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}$$

This can not be an equality unless the real parts are equal by themselves and also the imaginary parts. This determines two equations instead of one.

$$\begin{cases} \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \\ \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x} \end{cases}$$

These are the Cauchy-Riemann differential equations.

Because $\begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases}$

$$\Delta \xi = \frac{\partial \xi}{\partial x} \Delta x + \frac{\partial \xi}{\partial y} \Delta y$$

$$\Delta \eta = \frac{\partial \eta}{\partial x} \Delta x + \frac{\partial \eta}{\partial y} \Delta y$$

so that

$$\frac{\Delta \xi}{\Delta z} = \frac{\frac{\partial \xi}{\partial x} \Delta x + \frac{\partial \xi}{\partial y} \Delta y + i(\frac{\partial \eta}{\partial x} \Delta x + \frac{\partial \eta}{\partial y} \Delta y)}{\Delta x + i \Delta y}$$

If the necessary conditions are sufficient then the $\lim_{\Delta z \rightarrow 0} \frac{\Delta \xi}{\Delta z}$ will have a unique value no matter through what values Δz is allowed to approach zero. In other words Δz does not have to approach zero through either pure real or pure imaginary values but it may approach it through complex values $\Delta z = \Delta x + i \Delta y$, where Δx increments are entirely independent of Δy increments and neither need be zero.

Substituting the necessary conditions

$$\begin{cases} \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \\ \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x} \end{cases}$$

In the ^{last} equation

$$\begin{aligned}
 \frac{\Delta z}{\Delta z} &= \frac{\frac{\partial f}{\partial x} \cdot \Delta x - \frac{\partial g}{\partial x} \cdot \Delta y + i \left(\frac{\partial g}{\partial x} \Delta x + \frac{\partial f}{\partial x} \Delta y \right)}{\Delta x + i \Delta y} \\
 &= \frac{\frac{\partial f}{\partial x} \Delta x + i \frac{\partial f}{\partial x} \Delta y + i \frac{\partial g}{\partial x} \Delta y + i \frac{\partial g}{\partial x} \Delta x}{\Delta x + i \Delta y} \\
 &= \frac{\frac{\partial f}{\partial x} (\Delta x + i \Delta y) + i \frac{\partial g}{\partial x} (\Delta x + i \Delta y)}{(\Delta x + i \Delta y)} \\
 &= \left[\frac{\partial f}{\partial x} + i \frac{\partial g}{\partial x} \right] \frac{\Delta z}{\Delta z}
 \end{aligned}$$

$$\therefore \frac{dz}{dz} = \frac{\partial f}{\partial x} + i \frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} - i \frac{\partial f}{\partial y} = \frac{1}{i} \frac{\partial f}{\partial y} + \frac{\partial g}{\partial y} = \frac{1}{i} \left(\frac{\partial f}{\partial y} + i \frac{\partial g}{\partial y} \right)$$

Because the Δz cancels out the necessary conditions are therefore sufficient.

ANALYTIC FUNCTIONS.

Every function for which the Cauchy-Riemann differential equations are satisfied is called an analytic function.

Example:

Let $z = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$

$$\begin{cases} z = \xi + i\eta \\ z = x + iy \end{cases}$$

$$\xi + i\eta = e^x \cos y + i e^x \sin y$$

Equating real part of ξ to the real part of e^z and likewise the imaginary parts

$$\begin{cases} \xi = e^x \cos y \\ \eta = e^x \sin y \end{cases}$$

Test by applying the Cauchy-Riemann conditions

$$\begin{cases} \frac{\partial \xi}{\partial x} = e^x \cos y = \frac{\partial}{\partial y} (e^x \sin y) = \frac{\partial \eta}{\partial y} \\ \frac{\partial \xi}{\partial y} = -e^x \sin y = -\frac{\partial}{\partial x} (e^x \sin y) = -\frac{\partial \eta}{\partial x} \end{cases}$$

Therefore the function is analytic.

But not every function we may write will be analytic.

Let $\zeta = \xi + i\eta = x^2 - iy^2$

$$\begin{cases} \xi = x^2 \\ \eta = -y^2 \end{cases}$$

$$\frac{\partial \xi}{\partial x} = 2x$$

$$\frac{\partial \eta}{\partial y} = -2y$$

$$\frac{\partial \xi}{\partial x} \text{ is not equal } \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \xi}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial x} = 0$$

Therefore the derivative of this function has no unique value at any point in the plane and the function is not analytic.

SOME FACTS ABOUT A PARTICULAR TRANSFORMATION.

Let $\zeta = z^2$

where $\begin{cases} \zeta = \xi + i\eta \\ z = x + iy \end{cases}$

substituting

$$\zeta = \xi + i\eta = (x + iy)^2 = x^2 - y^2 + i2xy$$

Equating reals

$$f = x^2 - y^2$$

Equating imaginaries

Because
$$\left. \begin{aligned} \eta &= 2xy \\ \frac{\partial f}{\partial x} &= 2x = \frac{\partial \eta}{\partial y} \\ \frac{\partial f}{\partial y} &= -2y = -\frac{\partial \eta}{\partial x} \end{aligned} \right\} \text{the function is analytic.}$$

The complex values of z will be plotted in the z plane. The values of \mathfrak{z} determined by substituting values of z in the function ($\mathfrak{z} = z^2$) will be plotted in the \mathfrak{z} plane. A particular value of \mathfrak{z} , let it be \mathfrak{z} , gotten by substituting z , in the function is called the transform of z . The following graphs show some transformations from the z plane to the \mathfrak{z} plane.

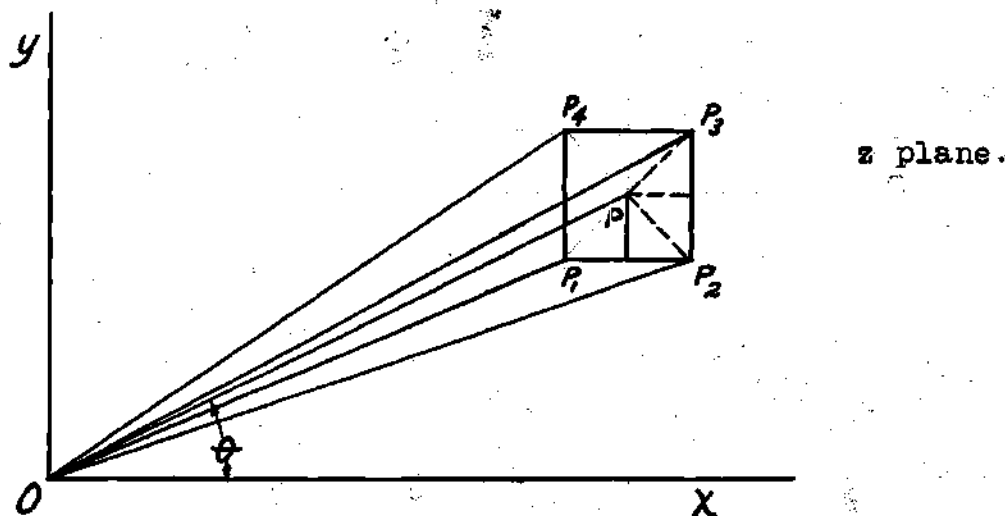


Fig. 31.

The square has each side = .01 and is shown exaggerated in the figure.

$$z_p = 2.005 + i 1.005 \text{ (angle with axis of reals } = \phi = 26^\circ - 37' - 20'')$$

$$z_{p_1} = 3 + i 1 \quad (\phi_1 = 26^\circ - 33' - 54'')$$

$$z_{p_2} = 2.01 + i 1 \quad (\phi_2 = 26^\circ - 27' - 3'')$$

$$z_{p_3} = 2.01 + i 1.01 \quad (\phi_3 = 26^\circ - 40' - 44'')$$

$$z_{p_4} = 2 + i 1.01 \quad (\phi_4 = 26^\circ - 47' - 37'')$$

Let the transforms of these be called $z_{p'}$, $z_{p'_1}$, $z_{p'_2}$, etc.

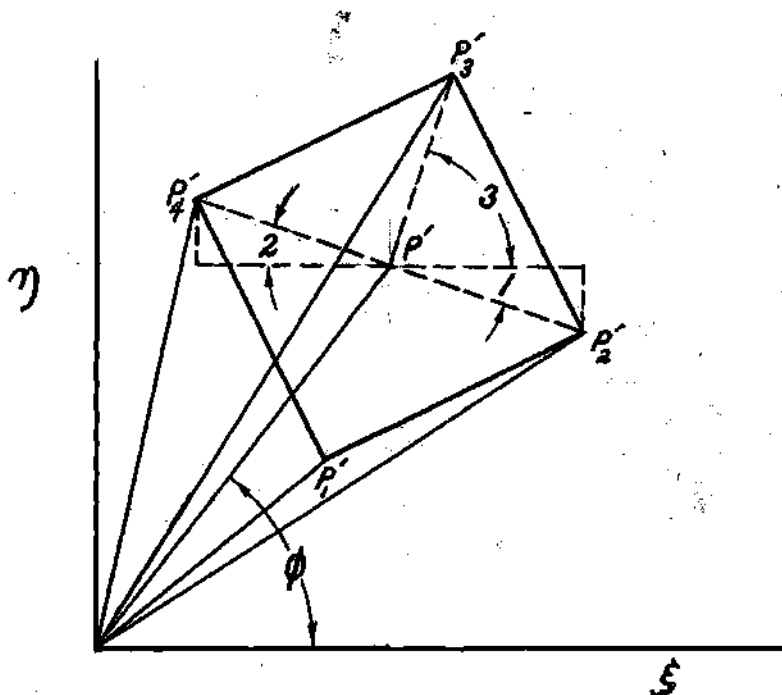
$$z_{p'} = \frac{2.005^2}{2.005^2 - 1.005^2} + i \frac{2 \times 2.005 \times 1.005}{2.005^2 - 1.005^2} = 3.01 + i 4.03 (\phi = 53^\circ - 14' - 40'')$$

$$z_{p'_1} = 3 + i 4 \quad (\phi_1 = 53^\circ - 7' - 48'')$$

$$z_{p'_2} = 3.04 + i 4.02 \quad (\phi_2 = 53^\circ - 54' - 6'')$$

$$z_{p'_3} = 3.02 + i 4.06 \quad (\phi_3 = 53^\circ - 21' - 28'')$$

$$z_{p'_4} = 2.98 + i 4.04 \quad (\phi_4 = 53^\circ - 35' - 14'')$$



z Plane.

Fig. 32.

The square $P'_1 P'_2 P'_3 P'_4$ is shown greatly exaggerated.

$$\tan \angle 1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4.02 - 4.03}{3.04 - 3.01} = -\frac{1}{3}$$

$$\tan \angle 2 = \frac{y_4 - y_1}{x_4 - x_1} = \frac{4.04 - 4.03}{2.98 - 3.01} = -\frac{1}{3}$$

Because $\tan \angle 1 = \tan \angle 2$ the points P'_2 , P'_1 and P'_4 are in the same straight line. Likewise the points P'_1 , P'_1 and P'_3 are in the same straight line.

$$\tan \angle 3 = \frac{y_3 - y_1}{x_3 - x_1} = \frac{4.06 - 4.03}{3.02 - 3.01} = +\frac{1}{3}$$

The slope of the line $P'_2 P'_1 P'_4 = -\frac{1}{3}$

The slope of the line $P'_1 P'_1 P'_3 = +\frac{1}{3}$.

These two lines are therefore perpendicular and the angle $P'_2 P'_1 P'_3$ is a right angle.

$$\begin{aligned} \text{The length of the line } P'_1 P'_2 &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3.04 - 3.01)^2 + (4.02 - 4.03)^2} \\ &= \sqrt{.04^2 + .02^2} \end{aligned}$$

$$\begin{aligned} \text{The length of the line } P'_1 P'_3 &= \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} \\ &= \sqrt{(3.02 - 3.01)^2 + (4.06 - 4.03)^2} \\ &= \sqrt{.04^2 + .02^2} \end{aligned}$$

The side $P'_1 P'_2$ is equal in length to the side $P'_1 P'_3$. Likewise all the sides are equal and the figure is a square.

The figure around the point P in the z plane is a square and its transform in the \mathcal{Z} plane is a square. The triangle $P_2 P P_3$ has been transformed into a similar triangle $P'_2 P'_1 P'_3$

because corresponding angles of the two triangles are equal. The lengths of corresponding sides are not equal but they are proportional. The angles of the other three triangles are also unchanged upon being transformed.

If any other triangle in the z plane with sides of elementary length and point P as one apex is transformed to the ζ plane the transform will be a similar triangle and the point P' in the ζ plane will be an apex. Such a transformation which preserves the angles of elementary figures is called Conformal.

An elementary triangle with sides drawn from O along OP and OP in the z plane transforms into the ζ plane with sides along OP' and OP' but the angle $P'O P'$ is double the angle POP . This transformation is not conformal and " O " is called a singular point. It can be demonstrated that every other point in the plane, excluding points where $z = \infty$, will produce a conformal transformation. All these details will be made clearer and more understandable in what follows.

The derivative of

$$\zeta = z^2$$

is

$$\frac{d\zeta}{dz} = 2z$$

$\frac{d\zeta}{dz}$ has a finite value at every point in the z plane except " O " and ∞ . It is always equal to $2z$ which makes it a complex number. Elementary lengths ($d\zeta$) drawn from a point P' in the ζ plane will be determined in absolute value and direction by

$$d\mathcal{Z} = \left(\frac{d\mathcal{Z}}{dz_p}\right) dz_p = (2z_p) dz_p$$

$$\text{Let } \begin{cases} z_p = OP \\ dz_p = PP_p \end{cases}$$

Fig. 31.

$$|z_p| = \sqrt{(2.005 + 1 \ 1.005)(2.005 - 1 \ 1.005)} = \sqrt{5.03}$$

$$\text{angle } \theta = 26^\circ - 37' - 20'' = .46464 \text{ radians}$$

$$|dz_p| = \sqrt{(2.005 - 2)^2 + (1.005 - 1)^2} = \sqrt{.000005}$$

$$\text{angle with } x = \tan^{-1} \frac{y_1 - y}{x_1 - x} = \tan^{-1} \frac{1 - 1.005}{2 - 2.005} = \tan^{-1} 1 \text{ [Third Quadrant]}$$

$$= \frac{5\pi}{4} \text{ radians} = 225^\circ$$

$$\text{which makes } \begin{cases} z_p = \sqrt{5.03} e^{i.46464} \\ dz_p = \sqrt{.000005} e^{i\frac{5\pi}{4}} \\ \left(\frac{d\mathcal{Z}}{dz_p}\right)_p = 2z_p = 2\sqrt{5.03} e^{i.46464} \end{cases}$$

Therefore

$$\begin{aligned} P'P'_1 &= d\mathcal{Z} = 2\sqrt{5.03} e^{i.46464} \sqrt{.000005} e^{i\frac{5\pi}{4}} \\ &= 2\sqrt{.000025} e^{i(.46464 + \frac{5\pi}{4})} \\ &= \sqrt{.001} e^{i(.46464 + \frac{5\pi}{4})} \end{aligned}$$

$$|d\mathcal{Z}| = \sqrt{.001}$$

$$\text{angle with } \xi \text{ axis} = 225^\circ + 26^\circ - 37' - 20'' = 251^\circ - 37' - 20''$$

From Fig. 32

$$\begin{aligned} |P'P'_1| &= |d\mathcal{Z}| = \sqrt{(\xi_1 - \xi)^2 + (\eta_1 - \eta)^2} = \sqrt{(3 - 3.01)^2 + (4 - 4.03)^2} \\ &= \sqrt{.001} \end{aligned}$$

$$\text{angle with } \xi \text{ axis} = \tan^{-1} \frac{\eta_1 - \eta}{\xi_1 - \xi} = \tan^{-1} \frac{-.03}{-.01} = 251^\circ - 33' - 54''$$

[The error in the angle is due to approximations in \mathcal{Z}_p and $\mathcal{Z}_{p'}$]

In the same manner

$$P'P'_2 = d\mathfrak{Z}_2 = \left(\frac{d\mathfrak{Z}}{dz}\right)_p dz_2$$

$$= 2\sqrt{5.03} e^{i.46464} \times \sqrt{.00005} e^{i\frac{7\pi}{4}}$$

$$|d\mathfrak{Z}_2| = \sqrt{.001}$$

$$\text{angle with } \mathfrak{Z} \text{ axis} = 315^\circ + 26^\circ - 37' - 20''$$

$$= 341^\circ - 37' - 20''$$

$$P'P'_3 = d\mathfrak{Z}_3 = \left(\frac{d\mathfrak{Z}}{dz}\right)_p dz_3$$

$$= 2\sqrt{5.03} e^{i.46464} \times \sqrt{.00005} e^{i\frac{\pi}{4}}$$

$$= \sqrt{.001} e^{i(\frac{\pi}{4} + .46464)}$$

$$P'P'_4 = d\mathfrak{Z}_4 = \sqrt{.001} e^{i(\frac{3\pi}{4} + .46464)}$$

It is evident that $\left\{ \begin{array}{l} \angle P_1 P P_2 = \angle P'_1 P' P'_2, \\ \angle P_2 P P_3 = \angle P'_2 P' P'_3 \\ \text{etc.} \end{array} \right\}$ which makes the

transformation conformal.

But suppose we wished to transform elementary lengths drawn from the origin in the z plane. For this case,

$$\frac{d\mathfrak{Z}}{dz} = 0$$

so

$$d\mathfrak{Z}_1 = 0, dz_1 = 0$$

This method breaks down where $\frac{d\mathfrak{Z}}{dz} = 0$ or where $\frac{d\mathfrak{Z}}{dz} = \infty$. For the first case a finite distance of dz in the z plane becomes a zero distance in the \mathfrak{Z} plane and for the second case a very small dz would transform to an infinite $d\mathfrak{Z}$. Points for which $\frac{d\mathfrak{Z}}{dz}$ are zero or infinity are called singular points and transformations made from those points are not conformal. But wherever the derivative is finite the transformation will

be conformal. If the origin in this case be excluded by a circle of infinitesimal radius the transformation at every point in the plane whose z co-ordinate is finite will be conformal.

The following facts have been illustrated by the preceding tedious and laborious method.

1. That if $z = f(z)$

$$\frac{dz}{dz} = f'(z) \text{ is also a complex number.}$$

2. If $f'(z)$ is not zero or infinity at a point the angle between any two lines drawn from the point will be preserved in the transformation.

3. Since $f'(z)$ and dz are both complex numbers their product will be another complex number dz the angle of which with the axis of reals will be the sum of the angles of the factors $f'(z)$ and dz . Because of this fact elementary figures around a point in the z plane will not occupy the same relative position to the axis of reals in the z plane as its transform in the z plane occupies relative to its axis of reals. In general then a transformation stretches or shortens an elementary length of the z plane to the corresponding elementary length of the z plane and rotates the latter with reference to the former. The square $P_1P_2P_3P_4$ (Fig.32) was rotated $26^\circ-37'-20''$ with reference to the square $P_1P_2P_3P_4$ (Fig.31) by the transformation.

4. The scale of the transformation at a point is the value of $f'(z)$ for that point. By this is meant that the value of $f'(z)$ stretches or shortens dz to make dz .

AN ELEMENTARY PROOF OF CONFORMAL TRANSFORMATION.

For a more rigorous proof see Wood's Advanced Calculus.

Let $\zeta = f(z)$ be a continuous function which has a derivative continuous for every point in the z plane.

Substitution of the z co-ordinate of point p in $f(z)$ determines the ζ co-ordinate of point p' .

The derivative of the function will be

$$\frac{d\zeta}{dz} = f'(z)$$

At the point p in the z plane

$$\text{Let } \begin{cases} f'(z) = ae^{i\alpha} \\ dz_1 = be^{i\beta} = pp_1 \\ dz_2 = ce^{i\gamma} = pp_2 \end{cases}$$

Then

$$d\zeta_1 = p'p'_1 = f'(z_p) \cdot (dz_1)_p = ae^{i\alpha} \cdot be^{i\beta} \\ = abe^{i(\alpha+\beta)}$$

$$d\zeta_2 = p'p'_2 = ae^{i\alpha} \cdot ce^{i\gamma} = ac \cdot e^{i(\alpha+\gamma)}$$

The angle $p, p_1, p_2 = (\beta - \gamma)$

The angle $p'_1, p', p'_2 = (\alpha + \beta) - (\alpha + \gamma) = (\beta - \gamma)$

Therefore the transformation is conformal.

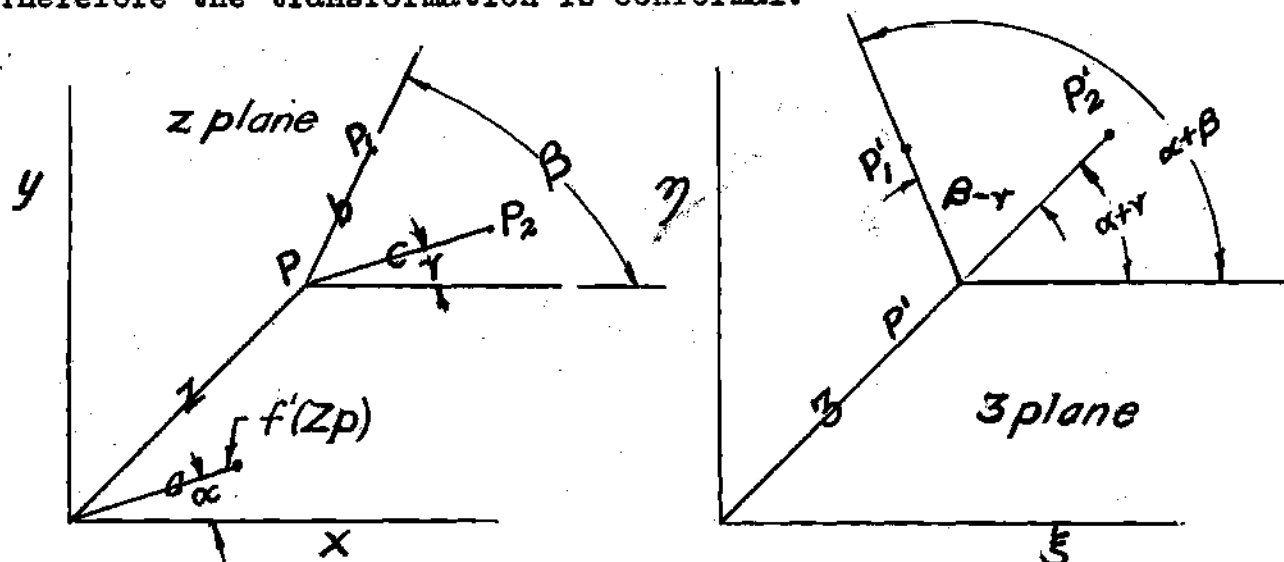


Fig. 33.

THE TRANSFORMATION OF A CIRCLE INTO A STRAIGHT LINE.

The Joukowski transformation is

$$\zeta = z + \frac{c^2}{z}$$

$$\begin{cases} \zeta = \xi + i\eta \\ z = x + iy \end{cases}$$

$$\xi + i\eta = x + iy + \frac{c^2}{x + iy} = x + iy + \frac{c^2(x - iy)}{x^2 + y^2}$$

$$\begin{cases} \xi = x + \frac{c^2 x}{x^2 + y^2} \\ \eta = y - \frac{c^2 y}{x^2 + y^2} \end{cases}$$

For every point on a circle of radius "c"

$$x^2 + y^2 = c^2.$$

The ξ and η co-ordinates of the transform of the circle in the ζ plane will be

$$\begin{cases} \xi = x + \frac{c^2 x}{c^2} = 2x \\ \eta = y - \frac{c^2 y}{c^2} = 0. \end{cases}$$

These are the co-ordinates of a straight line along the ξ axis in the ζ plane extending from $\xi = +2c$ to $\xi = -2c$.

Singular points of the transformation,

$$\frac{d\zeta}{dz} = 1 - \frac{c^2}{z^2}$$

By definition a singular point is one the z co-ordinate of which makes $\frac{d\zeta}{dz} = 0$ or ∞ . Equating the expression for

$$\frac{d\zeta}{dz} \text{ to } 0, \quad z^2 - c^2 = 0$$

$$z = \pm c$$

where $z=0$, $\frac{dz}{dz} = -\infty$. But this point is at the center of the circle and can not affect the transformation of points on the circumference of the circle.

$$z = \pm c \text{ where } \begin{cases} x = \pm c \\ y = 0 \end{cases}$$

The transformation is conformal, therefore, from a point on the circle where $z = ce^{i(\theta \pm \epsilon)}$ extending to $z = ce^{i(\pi \pm \epsilon)}$ where ϵ is an infinitesimal angle.

GEOMETRIC CONSTRUCTION FOR JOUKOWSKY TRANSFORMATION.

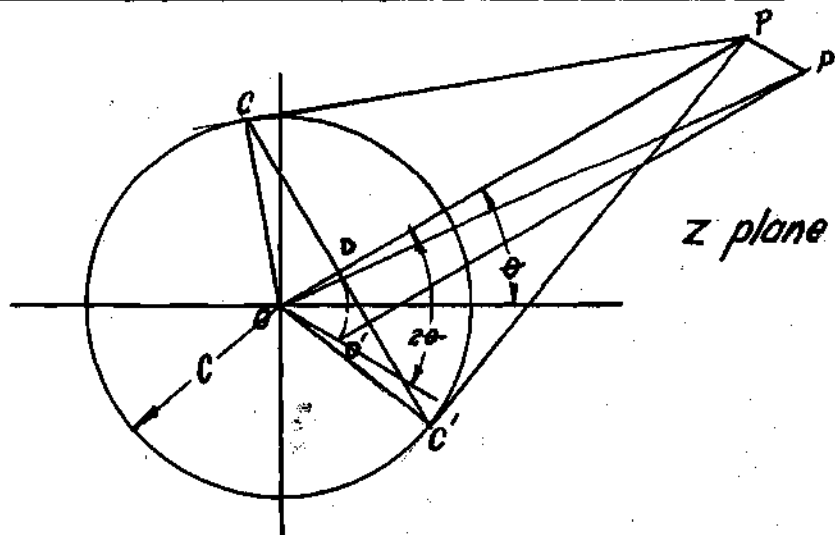


Fig. 34.

1. From point P draw the lines PC and PC' tangent to the circle of radius "C".
2. Draw the chord CC'.

Note that triangles OPC and OCD are right triangles with the $\angle COD$ common to both which makes them similar.

$$\cos COD = \frac{OD}{OC} = \frac{OC}{OP}$$

which makes

$$OD = \frac{OC^2}{OP} = \frac{c^2}{OP}$$

$$\text{If } OP = |z| = r$$

$$OD = \frac{1C^2}{|z|} = \frac{C^2}{|z|}$$

The length OD being the absolute value of $\frac{C^2}{z}$ we only have to get its angle to solve the problem. By use of the Euler form

$$\begin{aligned} \frac{C^2}{z} &= \frac{C^2}{re^{i\theta}} = \frac{C^2 e^{-i\theta}}{r} = \frac{C^2}{|z|} e^{-i\theta} = \frac{C^2}{|z|} (\cos\theta - i \sin\theta) \\ &= \frac{C^2}{|z|} \cos\theta - i \frac{C^2}{|z|} \sin\theta \end{aligned}$$

Therefore for the acute angle shown the real part is positive and the imaginary part negative. This puts $\frac{C^2}{z}$ in the fourth quadrant at the angle θ with the axis of reals. Therefore if OD is rotated through a negative angle of 2θ to OD' it will be in the correct position to represent $\frac{C^2}{z}$.

Finally addition of z to $\frac{C^2}{z}$ is effected by completing the parallelogram on OP and OD' making OP' represent in angle and direction

$$z = z + \frac{C^2}{z}$$

$\frac{C^2}{z}$ is determined by "inversion" on a circle of radius "C" and "reflection" in the axis of reals.

TRANSFORMATION OF A CIRCLE INTO AN AIRFOIL AND A PORTION OF THE CURVE OF EACH OF TWO STREAM LINES ABOUT THE CIRCLE TO THEIR CORRESPONDING POSITIONS ABOUT THE AIRFOIL.

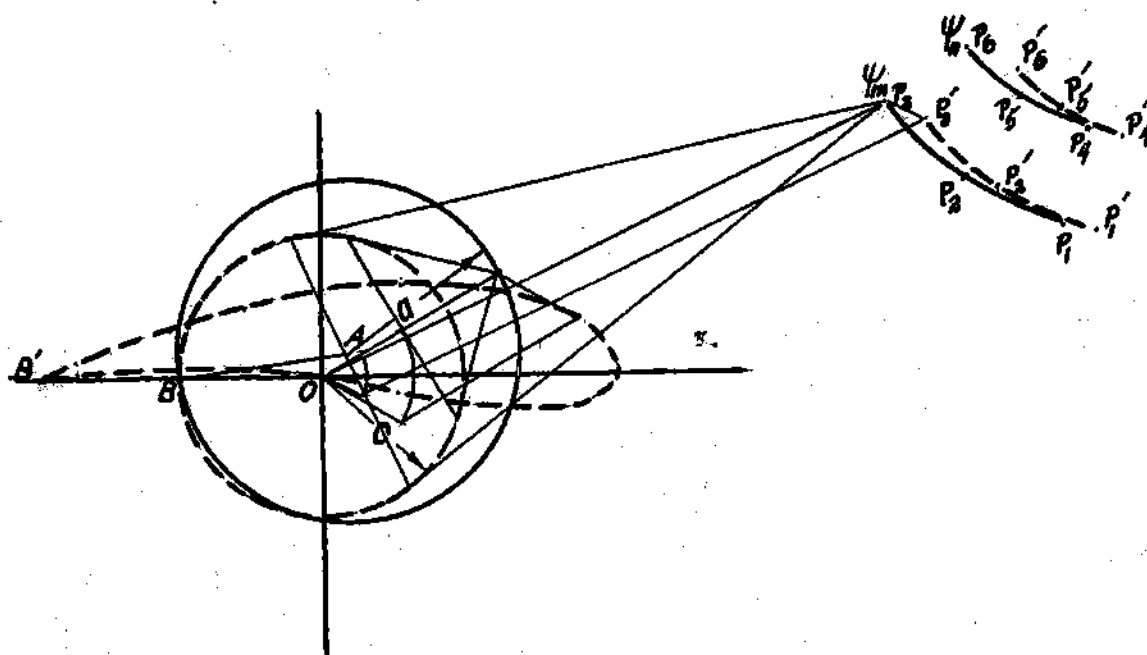


Fig. 35.

By the geometrical method a circle with center at A has been transformed to a closed curve figure shown with broken lines in the 3 plane. The 3 plane for convenience is shown on top of the z plane. This curved, closed figure bears a marked resemblance to an airfoil. By care in the selection of point A a very good airfoil shape may be determined. This is an illustration of how a circle in the z plane can be transformed into an airfoil in the 3 plane. For a more complete description of this see "Airfoil and Aircscrew Theory" by Pauert.

By the same method the curve of the stream line, ψ_m , between points P_1 and P_3 in the z plane has been transformed to the 3 plane and its transform is $P'_1 P'_2 P'_3$. Likewise the

length P_4 to P_6 of ψ_n has been transformed to $P'_4P'_5P'_6$. In this way all the stream lines of the flow about the circle could be transformed into those for the flow about the airfoil and the same transformation function would have to be used for both.

This is but a meager illustration of what is involved in the problem of transforming a circle and the flow about it into an airfoil with its corresponding flow. In what follows an attempt will be made to bring out all the hidden difficulties and a few of the ways around them.

THE POTENTIAL FUNCTION.

It has been proved that a function

$$\mathcal{F} = \xi + i\eta = f(z) = f(x + iy)$$

is analytic provided that

$$\frac{d\mathcal{F}}{dz} = f'(z) = \frac{\partial f}{\partial x} = \frac{\partial \xi}{\partial x} + i \frac{\partial \eta}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = \frac{1}{i} \left(\frac{\partial \xi}{\partial y} + i \frac{\partial \eta}{\partial y} \right)$$

which makes

$$\begin{cases} \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} \\ \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x} \end{cases}$$

Recalling that

$$\begin{cases} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = u \\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = v \end{cases}$$

It is evident that ϕ and ψ can be made into an analytic function. Call this function w and write

$$w = \phi + i\psi$$

The value of w for a particular flow is called the potential function of that flow. There is no connection between the w representing the potential function and the symbol, used for lack of a better one, in such expressions as $e^w = z$, etc.

A FEW POTENTIAL FUNCTIONS.

For a uniform flow parallel to the axis of x :

$$\phi = Ux, \quad \psi = Uy$$

$$\text{since } w = \phi + i\psi$$

$$w = U(x + iy) = Uz$$

Uniform flow parallel to the axis of y :

$$\phi = Vy, \quad \psi = -Vx$$

$$\therefore w = V(y - ix) = -iV(x + iy) = -iVz$$

Source at the origin:

$$\phi = \frac{m}{2\pi} \log r, \quad \psi = \frac{m}{2\pi} \theta$$

$$\begin{aligned} \therefore w &= \frac{m}{2\pi} (\log r + i\theta) = \frac{m}{2\pi} (\log r + i\theta \log e) \\ &= \frac{m}{2\pi} \log r e^{i\theta} = \frac{m}{2\pi} \log z \end{aligned}$$

Doublet at the origin with axis along the axis of x :

$$\phi = -\frac{\mu}{2\pi} \cdot \frac{x}{r^2}, \quad \psi = \frac{\mu}{2\pi} \cdot \frac{y}{r^2}$$

$$\begin{aligned} \therefore w &= \frac{\mu}{2\pi r^2} (-x + iy) = -\frac{\mu}{2\pi r^2} (x - iy) = -\frac{\mu}{2\pi r} \left(\frac{x}{r} - i \frac{y}{r} \right) \\ &= -\frac{\mu}{2\pi r} (\cos \theta - i \sin \theta) = -\frac{\mu}{2\pi r} e^{-i\theta} = -\frac{\mu}{2\pi r e^{i\theta}} = -\frac{\mu}{2\pi z} \end{aligned}$$

Flow parallel to the negative branch of the axis of x with circulation past a circle of radius a with center at the origin:

$$\phi = -U\left(x + \frac{a^2 \cos \theta}{r}\right) + \frac{K}{2\pi} \theta, \quad \psi = -U\left(y - \frac{a^2 \sin \theta}{r}\right) - \frac{K}{2\pi} \log \frac{r}{a}$$

$$\therefore W = -U\left(x + \frac{a^2 \cos \theta}{r}\right) - iU\left(y - \frac{a^2 \sin \theta}{r}\right) + \frac{K}{2\pi} (\theta - i \log \frac{r}{a})$$

$$= -U(x + iy) - a^2 U \left(\frac{\cos \theta}{r} - \frac{i \sin \theta}{r} \right) - i \frac{K}{2\pi} \left(i\theta + \log \frac{r}{a} \right)$$

$$= -Uz - \frac{a^2 U}{z} - \frac{iK}{2\pi} \log_e \left(e^{i\theta} \cdot \frac{r}{a} \right)$$

$$= -U \left(z + \frac{a^2}{z} \right) - \frac{iK}{2\pi} \log_e \frac{z}{a}$$

THE DETERMINATION OF THE VELOCITY AT EVERY POINT IN THE z

PLANE.

If the components of velocity parallel to the axes, ξ and η , are known the velocity is determined.

Before these components at any point in the plane can be determined the transformation function must have two very definite characteristics:

1. The function must be analytic. Which means that if

$$z = \xi + i\eta = f(z) = f(x + iy)$$

$$\frac{dz}{dz} = f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$$

This makes the transformation conformal.

2. $z = z$, where $z = \infty$.

This means that both planes are identical at infinity.

It is evident from the form of

$$z = z^2$$

that it does not meet this requirement, although it does meet the first. On the other hand Joukowski's transformation

$$z = z + \frac{c^2}{z}$$

becomes for $z = \infty$

$$\mathcal{Z} = z.$$

Further along there will appear other transformations but inspection will show that in each case both these conditions are fulfilled.

The necessity for this last condition requires some discussion. In Fig. 35 the graphical transformation of two stream lines, ψ_m and ψ_n , was illustrated. As the absolute value of z gets larger the absolute value of \mathcal{Z} becomes more nearly equal to it. This means that if the \mathcal{Z} plane be placed on top of the z plane it will be noted that the stream lines in the \mathcal{Z} plane will approach nearer and nearer the same form as those in the z plane as the absolute value of z increases until at remote distances, $|z| = \infty$, the stream lines for both planes will match identically and become indistinguishable. The quantity of fluid, $\psi_m - \psi_n$, which is flowing per unit time between the stream lines ψ_m and ψ_n at infinity in the z plane will therefore be diverted by the transformation without change and caused to flow between the transformed streamlines in the \mathcal{Z} plane. This very definitely connects the velocities in the two planes. The transformation serves as a two way valve at infinity, between any two streamlines, for diverting the fluid from the z plane to the \mathcal{Z} plane. This makes the streamline ψ_m for the z plane and its transform in the \mathcal{Z} plane have the same value of ψ . Because ψ_m is any stream line. The same reasoning holds for the lines of

constant velocity potential.

In Fig. 35 $P_1 P_4$ was drawn as the normal to streamlines ψ_m and ψ_n in the z plane so that $P'_1 P'_4$ would be normal to the streamlines ψ_m and ψ_n in the ζ plane.

Let $\psi_m - \psi_n = d\psi$.

Then because $Q = A V$

$$d\psi = q \times \overline{P_1 P_4} = q' \times \overline{P'_1 P'_4}$$

$$q' = q \cdot \frac{\overline{P_1 P_4}}{\overline{P'_1 P'_4}} = q \left| \frac{dz}{d\zeta} \right|_{P=P'}$$

$q' =$ Resultant velocity at point P' in ζ plane.

The velocities at corresponding points in the two planes are evidently inversely proportional to the corresponding elementary normals to the stream lines at those points. If an elementary normal in the z plane is made longer by the transformation the corresponding velocity in the ζ plane becomes smaller and vice-versa.

Because of all this the potential function, w , becomes a function of both z and ζ .

$$w = \phi(z) = F(\zeta) = \phi + i\psi.$$

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} = \frac{1}{i} \frac{\partial w}{\partial y} = \frac{1}{i} \left(\frac{\partial \phi}{\partial y} + i \frac{\partial \psi}{\partial y} \right)$$

$$= u - iv$$

$$\frac{dw}{d\zeta} = \frac{\partial w}{\partial \xi} = \frac{\partial \phi}{\partial \xi} + i \frac{\partial \psi}{\partial \xi} = \frac{1}{i} \frac{\partial w}{\partial \eta} = \frac{1}{i} \left(\frac{\partial \phi}{\partial \eta} + i \frac{\partial \psi}{\partial \eta} \right)$$

$$= u' - iv'$$

where $u' =$ component of velocity in ζ plane parallel to ξ axis.

where $v' =$ component of velocity in \mathcal{Z} plane parallel to η axis.

In the z plane at point P,

$$d\psi = udy - vdx$$

$$d\phi = udx + vdy.$$

In the \mathcal{Z} plane at point P', the transform of point P,

$$d\psi = u'd\eta - v'd\xi$$

$$d\phi = u'd\xi + v'd\eta$$

But

$$dw = d\phi + id\psi$$

And from the twin expressions for $d\phi$ and likewise for $d\psi$ we may write

$$\begin{aligned} [u'd\xi + v'd\eta] + i[-v'd\xi + u'd\eta] &= [udx + vdy] + i[-vdx + udy] \\ u'[d\xi + id\eta] + v'[d\eta - id\xi] &= u[dx + idy] + v[dy - idx] \\ u'd\mathcal{Z} - iv'\left[d\xi - \frac{d\eta}{i}\right] &= udx - iv\left[dx - \frac{dy}{i}\right] \end{aligned}$$

but

$$\frac{-1}{i} = \frac{-1i}{i^2} = \frac{-1i}{-1} = i$$

so

$$\begin{cases} d\xi - \frac{d\eta}{i} = d\xi + id\eta = d\mathcal{Z} \\ dx - \frac{dy}{i} = dx + idy = dz \end{cases}$$

substituting

$$\begin{aligned} [u' - iv'] d\mathcal{Z} &= [u - iv] dz \\ u' - iv' &= [u - iv] \frac{dz}{d\mathcal{Z}} \end{aligned}$$

but
$$\begin{cases} u - iv = \frac{dw}{dz} \\ \text{and likewise} \\ u' - iv' = \frac{dw}{d\zeta} \end{cases}$$

therefore,

$$\frac{dw}{d\zeta} = u' - iv' = \frac{dw}{dz} \frac{dz}{d\zeta}$$

Or by equating the real parts on each side of the equation and likewise the imaginaries

$$\begin{aligned} u' &= u \left| \frac{dz}{d\zeta} \right| \\ v' &= v \left| \frac{dz}{d\zeta} \right| \end{aligned}$$

Squaring and adding

$$u'^2 + v'^2 = q'^2 = (u^2 + v^2) \left| \frac{dz}{d\zeta} \right|^2 = q^2 \left| \frac{dz}{d\zeta} \right|^2$$

or
$$q' = q \left| \frac{dz}{d\zeta} \right|$$

Therefore the analytical method checks the graphical for the equation for the transformation of velocity magnitudes.

THE TRANSFORMATION OF A NON-CIRCULATORY, RECTILINEAR FLOW PARALLEL TO THE AXIS OF REALS IN THE ζ PLANE TO THE z PLANE.

We know that the stream function in the z plane ^{for a} non-circulatory, rectilinear flow parallel to the x axis and directed in the negative sense is

$$\psi = -Uy.$$

Likewise the velocity potential for the same flow is

$$\phi = -Ux$$

Without derivation and directly by analogy this type of flow in the ζ plane will be represented by,

$$\psi = -U\eta$$

$$\phi = -U\xi$$

$$w = \phi + i\psi = -U(\xi + i\eta) = -U\zeta$$

Joukowski's transformation is

$$\zeta = z + \frac{c^2}{z}$$

Substituting

$$z + \frac{c^2}{z}$$

for ζ in w gives the potential function in the z plane

$$w = -U\left(z + \frac{c^2}{z}\right).$$

This is the potential function for a flow which is uniform and parallel to the x axis at infinity about a circle of radius " c " with center at origin. Previously this function was derived by a far more laborious method.

This same transformation transforms that part of the ξ axis extending from $-2c$ to $+2c$ into the circle of radius " c " in the z plane as has previously been shown. The feature of the transformation which should be noted at this point is the relationship between the velocity in the ζ plane at the points, $(-2c, 0)$ and $(2c, 0)$, and the velocity in the z plane at the transform of those points. Substituting the points, $(\pm 2c, 0)$ in

$$\zeta = \xi + i\eta = z + \frac{c^2}{z} = x + iy + \frac{c^2}{x + iy}$$

gives

$$\pm 2c + i0 = x + iy + \frac{c^2}{(x^2 + y^2)}x - \frac{ic^2 y}{x^2 + y^2}$$

Equating the imaginaries

$$y \left[\frac{x^2 + y^2 - c^2}{x^2 + y^2} \right] = 0$$

which gives

$$x^2 + y^2 = c^2$$

Equating the reals

$$x \left[\frac{x^2 + y^2 + c^2}{x^2 + y^2} \right] = \pm 2c$$

Substituting c^2 for $x^2 + y^2$

$$x \frac{2c^2}{c^2} = \pm 2c$$

$$x = \pm c$$

$$y = 0$$

At $(\pm c, 0)$ in the z plane

$$\frac{d\zeta}{dz} = 1 - \frac{c^2}{z^2} = 0$$

Now because $q' = q \left| \frac{dz}{d\zeta} \right|$

$$q = q' \left| \frac{d\zeta}{dz} \right|$$

At the points $(\pm 2c, 0)$ in the ζ plane

$$u' = -\frac{\partial \psi}{\partial \zeta} = -U = q'$$

$$v' = -\frac{\partial \psi}{\partial \bar{\zeta}} = 0$$

Which makes the velocities in the z plane at the points $(\pm c, 0)$

$$q = -U \left| \frac{dz}{dz} \right|_{\substack{z = \pm 2c \\ \eta = 0}} = -U \cdot 0 = 0$$

Suppose the transformation had been from the z plane to the ζ plane. The points $(\pm c, 0)$ would have transformed to the points $(\pm 2c, 0)$. The velocities at these points are gotten from

$$u = \frac{\partial \psi}{\partial y}$$

$$v = - \frac{\partial \psi}{\partial x}$$

where $w = \phi + i\psi = -U \left(z + \frac{c^2}{z} \right) = -U \left(x + iy + \frac{c^2 x}{x^2 + y^2} - \frac{ic^2 y}{x^2 + y^2} \right)$

Equating imaginaries

$$\begin{aligned} \psi &= -Uy \left(1 - \frac{c^2}{x^2 + y^2} \right) \\ \frac{\partial \psi}{\partial y} &= -U \left(1 - \frac{c^2}{x^2 + y^2} \right) - Uy \left[\frac{2c^2 y}{(x^2 + y^2)^2} \right] = u \\ -\frac{\partial \psi}{\partial x} &= +Uy \left[\frac{2c^2 x}{x^2 + y^2} \right] = v \end{aligned}$$

Substituting

$$x = \pm c$$

$$y = 0$$

$$u = v = 0 = q$$

These points are called stagnation points.

$$\left| \frac{dz}{dz} \right|_{\substack{x = \pm c \\ y = 0}} = 0$$

which makes

$$\left| \frac{dz}{dz} \right|_{\substack{x = \pm c \\ y = 0}} = \frac{1}{0} = \infty$$

which makes

$$q' = q \left| \frac{dz}{d\zeta} \right|_{\substack{x=\pm c \\ y=0}} = 0 \cdot \infty \text{ [an indeterminate quantity]}$$

But we know that in this case

$$0 \cdot \infty = -U$$

The points in the z plane ($\pm c, 0$) are singular points. If q has a finite value at such points

$$q' = q \cdot \infty = \infty$$

Therefore in making a transformation from the z plane to the ζ plane the singular points in the z plane must also be made stagnation points of the flow if infinite velocities in the ζ plane are to be avoided.

THE TRANSFORMATION OF A NON-CIRCULATORY FLOW ABOUT A CIRCLE (PROJECTION OF A CYLINDER) WITH CENTER AT ORIGIN IN Z PLANE TO A FLOW ABOUT A LINE (PROJECTION OF A PLANE) ALONG AXIS OF REALS IN ζ PLANE. THE FLOW AT INFINITY IN BOTH PLANES IS RECTILINEAR AND MAKES AN ANGLE OF α WITH THE AXIS OF REALS. THE TRANSFORMATION TO BE EFFECTED BY JOUKOWSKI'S METHOD.

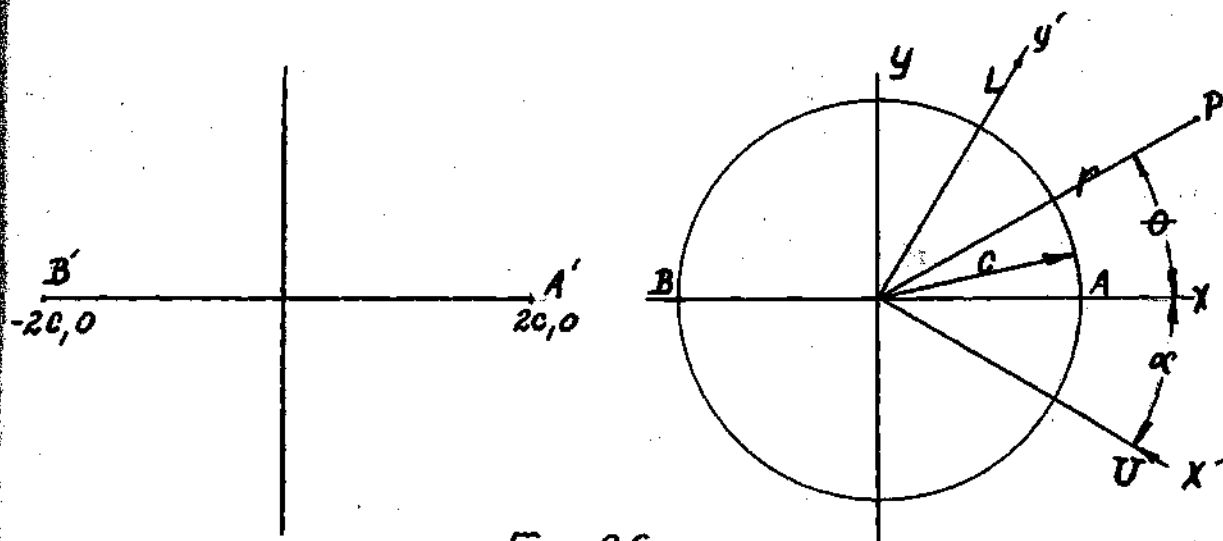


Fig. 36.

Let the axis parallel to U be x' and the complex co-ordinate for the $[x', y']$ plane be z' .

The z co-ordinate of point P is

$$z = re^{i\theta}$$

The z' co-ordinate of the point P is

$$z' = re^{i(\theta + \alpha)}$$

The relationship between the co-ordinates is obtained by division and is

$$\frac{z'}{z} = \frac{re^{i(\theta + \alpha)}}{re^{i\theta}} = e^{i(\theta + \alpha - \theta)} = e^{i\alpha}$$

$$z' = ze^{i\alpha}$$

The potential function for the z' plane is by analogy

$$w = -U\left(z' + \frac{c^2}{z'}\right)$$

By substituting $ze^{i\alpha}$ for z' , the potential function for the z plane is

$$w = -U\left(ze^{i\alpha} + \frac{c^2}{ze^{i\alpha}}\right)$$

$$\frac{dw}{dz} = -U\left(e^{i\alpha} - \frac{c^2 e^{-i\alpha}}{z^2}\right)$$

The transformation function is

$$\zeta = z + \frac{c^2}{z}$$

so that

$$\frac{d\zeta}{dz} = 1 - \frac{c^2}{z^2}$$

and

$$\frac{dz}{d\zeta} = \frac{z^2}{z^2 - c^2}$$

From all this

$$\frac{dw}{dz} = u' - iv' = \frac{dw}{dz} \frac{dz}{dz} = -U \left(\frac{z^2 e^{i\alpha} - c^2 e^{-i\alpha}}{z^4} \right) \left(\frac{z^2}{z^2 - c^2} \right)$$

Substituting $re^{i\theta}$ for z

$$\begin{aligned} \frac{dw}{dz} &= -U \left[\frac{cr^2 e^{i2\theta} e^{i\alpha}}{c} - \frac{c^2 re^{i\theta} e^{-i\alpha}}{re^{i\theta}} \right] \left[\frac{1}{\frac{cr^2 e^{i2\theta}}{c} - \frac{c^2 re^{i\theta}}{re^{i\theta}}} \right] \\ &= -U \cancel{re^{i\theta}} \left[\left(\frac{r}{c} \right) e^{i\theta} e^{i\alpha} - \left(\frac{c}{r} \right) \frac{e^{-i\alpha}}{e^{i\theta}} \right] \frac{1}{\cancel{cre^{i\theta}} \left[\frac{r}{c} e^{i\theta} - \frac{c}{r} \frac{1}{e^{i\theta}} \right]} \end{aligned}$$

$$\begin{cases} e^{i\theta} e^{i\alpha} = e^{i(\theta+\alpha)} \\ \frac{c}{r} = \left(\frac{r}{c} \right)^{-1} \\ \frac{e^{-i\alpha}}{e^{i\theta}} = e^{-i\alpha} \cdot e^{-i\theta} = e^{-i(\theta+\alpha)} \end{cases}$$

Let $\frac{r}{c} = e^\lambda$

then

$$\frac{dw}{dz} = -U \left[e^{\lambda+i(\theta+\alpha)} - e^{-\lambda-i(\theta+\alpha)} \right] \cdot \frac{1}{e^{\lambda+i\theta} - e^{-\lambda-i\theta}}$$

By definition

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

Hence

$$\begin{aligned} e^{\lambda+i(\theta+\alpha)} - e^{-\lambda-i(\theta+\alpha)} &= 2 \sinh [\lambda+i(\theta+\alpha)] \\ e^{\lambda+i\theta} - e^{-\lambda-i\theta} &= 2 \sinh [\lambda+i\theta] \end{aligned}$$

which makes

$$\frac{dw}{dz} = - \frac{2U \sinh [\lambda+i(\theta+\alpha)]}{2 \sinh [\lambda+i\theta]}$$

Expanding

$$\begin{aligned}\sinh [\lambda + i(\theta + \alpha)] &= \sinh \lambda \cosh [i(\theta + \alpha)] \\ &\quad + \cosh \lambda \sinh [i(\theta + \alpha)] \\ \sinh [\lambda + i\theta] &= \sinh \lambda \cosh i\theta + \cosh \lambda \sinh i\theta\end{aligned}$$

By definition

$$\begin{aligned}\cosh [i(\theta + \alpha)] &= \frac{e^{i(\theta + \alpha)} + e^{-i(\theta + \alpha)}}{2} = \cos (\theta + \alpha) \\ \sinh [i(\theta + \alpha)] &= \frac{e^{i(\theta + \alpha)} - e^{-i(\theta + \alpha)}}{2} = i \sin (\theta + \alpha)\end{aligned}$$

so that

$$\begin{aligned}\sinh [\lambda + i(\theta + \alpha)] &= \sinh \lambda \cos (\theta + \alpha) + \\ &\quad i \cosh \lambda \sin (\theta + \alpha) \\ \sinh [\lambda + i\theta] &= \sinh \lambda \cos \theta + i \cosh \lambda \sin \theta\end{aligned}$$

Futhermore,

$$\begin{aligned}\cos (\theta + \alpha) &= \cos \theta \cos \alpha - \sin \theta \sin \alpha \\ \sin (\theta + \alpha) &= \sin \theta \cos \alpha + \cos \theta \sin \alpha\end{aligned}$$

Making a double substitution,

$$\frac{dw}{dz} = -U \frac{\sinh \lambda (\cos \theta \cos \alpha - \sin \theta \sin \alpha) + i \cosh \lambda (\sin \theta \cos \alpha + \cos \theta \sin \alpha)}{\sinh \lambda \cos \theta + i \cosh \lambda \sin \theta}$$

Multiplying numerator and denominator by

$$\sinh \lambda \cos \theta - i \cosh \lambda \sin \theta$$

$$\begin{aligned}\frac{dw}{dz} &= -U \left[\frac{\sinh^2 \lambda (\cos^2 \theta \cos \alpha - \sin \theta \cos \theta \sin \alpha) + \cosh^2 \lambda (\sin^2 \theta \cos \alpha + \sin \theta \cos \theta \sin \alpha)}{\sinh^2 \lambda \cos^2 \theta + \cosh^2 \lambda \sin^2 \theta} \right] \\ &\quad - iU \left[\frac{\sinh \lambda \cosh \lambda (\sin \theta \cos \theta \cos \alpha + \cos^2 \theta \sin \alpha) - \sinh \lambda \cosh \lambda (\sin \theta \cos \theta \cos \alpha - \sin^2 \theta \sin \alpha)}{\sinh^2 \lambda \cos^2 \theta + \cosh^2 \lambda \sin^2 \theta} \right]\end{aligned}$$

By definition

$$\begin{aligned}\cosh z &= \frac{e^z + e^{-z}}{2} \\ \sinh z &= \frac{e^z - e^{-z}}{2}\end{aligned}$$

squaring $\cosh z$ and subtracting $\sinh^2 z$

$$\cosh^2 z - \sinh^2 z = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1$$

or $\cosh^2 z = 1 + \sinh^2 z$

Substituting $1 + \sinh^2 z$ for $\cosh^2 z$ in the denominator of $\frac{dw}{dz}$

it becomes

$$\sinh^2 \lambda \cos^2 \theta + \sin^2 \theta + \sinh^2 \lambda \sin^2 \theta = \sinh^2 \lambda + \sin^2 \theta$$

Remembering that

$$\frac{dw}{dz} = u' - iv'$$

By equating the reals of both sides

$$u' = -U \left[\frac{\sinh^2 \lambda (\cos^2 \theta \cos \alpha - \sin^2 \theta \cos \theta \sin \alpha) + \cosh^2 \lambda (\sin^2 \theta \cos \alpha + \sinh^2 \lambda \sin^2 \theta \sin \alpha)}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

By equating the imaginaries of both sides

$$v' = U \left[\frac{\sinh \lambda \cosh \lambda (\sin \theta \cos \theta \cos \alpha + \cos^2 \theta \sin \alpha) - \sinh \lambda \cosh \lambda (\sin^2 \theta \cos \alpha - \sin^2 \theta \sin \alpha)}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$u' = -U \left[\frac{\sinh^2 \lambda \cos^2 \theta \cos \alpha - \sinh^2 \lambda \sin^2 \theta \cos \theta \sin \alpha + \cosh^2 \lambda \sin^2 \theta \cos \alpha + \cosh^2 \lambda \sin^2 \theta \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$= -U \left[\frac{\sinh^2 \lambda \cos^2 \theta \cos \alpha + (1 + \sinh^2 \lambda) \sin^2 \theta \cos \alpha + (\cosh^2 \lambda - \sinh^2 \lambda) \sin^2 \theta \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$= -U \left[\frac{\sin^2 \theta \cos \alpha + \sinh^2 \lambda (\cos^2 \theta + \sin^2 \theta) \cos \alpha + \sin^2 \theta \cos \theta \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$= -U \left[\cos \alpha + \frac{\sin \theta \cos \theta \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$v' = U \left[\frac{\sinh \lambda \cosh \lambda \sin \theta \cos \theta \cos \alpha + \sinh \lambda \cosh \lambda \cos^2 \theta \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta} + \right.$$

$$\left. - \frac{\sinh \lambda \cosh \lambda \sin \theta \cos \theta \cos \alpha + \sinh \lambda \cosh \lambda \sin^2 \theta \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$= U \left[\frac{\sinh \lambda \cosh \lambda \sin \alpha (\cos^2 \theta + \sin^2 \theta)}{\sinh^2 \lambda + \sin^2 \theta} \right]$$

$$= U \frac{\sinh \lambda \cosh \lambda \sin \alpha}{\sinh^2 \lambda + \sin^2 \theta}$$

By reference to Fig. 36 the circle of radius "c" in the z plane transforms to a line from $(-2c, 0)$ to $(+2c, 0)$ on the ξ axis in the ξ plane. If we want to obtain the velocities at different points on this line we must confine our points in the z plane to those on the circle of radius "c". For this circle

$$\frac{r}{c} = \frac{0}{c} = 1 = e^0$$

Which makes $\lambda = 0$

Putting this value of λ in the equations for u' and v' and remembering that

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$$

$$u' = -U \left[\cos \alpha + \frac{\sinh \theta \cos \theta \sin \alpha}{\sin^2 \theta} \right] = -U \left[\cos \alpha + \cot \theta \sin \alpha \right]$$

$$v' = 0 \text{ except where } \theta = 0^\circ \text{ or } 180^\circ$$

At the singular points

$$\theta = 0^\circ \text{ or } 180^\circ$$

Therefore, because $\cot 0 = \infty$ and $\cot \pi = -\infty$

$$u' = \pm \infty$$

$$v' = \frac{0}{0} \text{ [Indeterminate quantity]}$$

For any circle other than the one the radius of which is "c", v' is a finite determinate quantity. Where θ is 0° or 180° , the substitution of $\lambda = 0$ produces the indeterminate expression. To evaluate v' by L'Hospital's rule the derivative must be taken with respect to λ . Before evaluating, the differentiation of hyperbolic functions will be illustrated.

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\frac{d}{dz} [\sinh z] = \frac{e^z - e^{-z}(-1)}{2} = \frac{e^z + e^{-z}}{2} = \cosh z$$

$$\frac{d}{dz} [\cosh z] = \frac{e^z + e^{-z}(-1)}{2} = \frac{e^z - e^{-z}}{2} = \sinh z.$$

Taking independently the derivative with respect to λ of the numerator and denominator of the expression for v'

$$v' = U \sin \alpha \left[\frac{\sinh \lambda \sinh \lambda + \cosh \lambda \cosh \lambda}{2 \sinh \lambda \cosh \lambda + \sin^2 \theta} \right]$$

Let $\lambda \rightarrow 0$

$$\theta = 0$$

$$\begin{aligned} v' &= U \sin \alpha \left[\frac{0+1}{2 \cdot 0 \cdot 1 + 0} \right] \\ &= U \sin \alpha \left[\frac{1}{0} \right] = \infty. \end{aligned}$$

For this type of flow, therefore, there is a finite value of q at the singular points which makes

$$q' = q \cdot \infty = \infty.$$

The special case of a flow perpendicular to a line [more exactly a plane, 4c wide and infinitely long] is obtained by substituting $\pm \frac{\pi}{2}$ for α but strange to say only very viscous slow moving liquids will move in accordance with the stream lines of this theoretical case. Ignoring the two points of infinite velocity Glauert makes practical use of this case in his analysis of velocities in the vicinity of a wing.

THE TRANSFORMATION OF $w = -U(z' + \frac{c^2}{z'}) - i \frac{k}{2\pi} \log \frac{z'}{c}$

TO THE FLOW ABOUT THE TRANSFORM OF THE CIRCLE OF RADIUS "C" IN THE z' PLANE BY USE OF JOUKOWSKI'S TRANSFORMATION.

From Fig. 36

$$z' = ze^{i\alpha}$$

Substituting $ze^{i\alpha}$ for z' in the expression for w ,

$$w = -U\left(ze^{i\alpha} + \frac{c^2}{ze^{i\alpha}}\right) - i \frac{k}{2\pi} \log \frac{ze^{i\alpha}}{c}$$

$$\frac{dw}{dz} = -U\left(e^{i\alpha} - \frac{c^2 e^{-i\alpha}}{z^2}\right) - i \frac{k}{2\pi} \frac{1}{z}$$

By use of this last equation and Joukowski's hypothesis a unique value of k can be determined. An explanation of Joukowski's hypothesis follows.

JOUKOWSKI'S HYPOTHESIS.

In Fig. 35. the point B on the circle of radius "a" has the complex co-ordinate $z = ce^{i\pi}$. This is a value of z which makes

$$\frac{d\delta}{dz} = \left(1 - \frac{c^2}{z^2}\right) = \left(1 - \frac{c^2}{c^2}\right) = 0$$

and of course point B becomes a singular point of the transformation. This point transforms to the trailing edge (which is a cusp) of the airfoil. The external angle to the circle at point B is π while the external angle to the airfoil at B' is 2π . If for every angle of attack the circulation be adjusted so that point B is a stagnation point then the velocity at point B', although indeterminate, will not be infinity

$$\left[q'\right]_{B'} = \left[q\right]_B \cdot \infty = 0 \cdot \infty.$$

This permits the air to leave the airfoil smoothly at the trailing edge. Since most airfoils have a rather sharp trailing edge this is a good practical hypothesis.*

*See "Aerofoil and Airscrew Theory" by Glauert for a critical discussion of this hypothesis.

The application of Joukowski's hypothesis to the case under discussion will eliminate the infinite velocity at the trailing edge but not at the leading edge. This case is brought in to the discussion because of this very reason. Glauert's analysis of the lift of thin airfoils has the defect of having an infinite velocity at the leading edge and the reason why is apparent after the discussion of the present case.

From Fig. 36 and all that has preceded, it is apparent that point B whose complex co-ordinate is $ce^{i\pi}$ transforms to point B', the trailing edge of the line [more exactly a flat plate, $4c$ wide and infinitely long.] By Joukowski's hypothesis point B must be a stagnation point on the circle in plane z for every value of the angle of attack, α .

Because

$$\frac{dw}{dz} = u - iv$$

and at a stagnation point $u = v = 0$, for point B we must make

$$u - iv = 0 = \frac{dw}{dz}.$$

Therefore, substituting $z = ce^{i\pi}$

$$\begin{aligned} \frac{dw}{dz} = 0 &= -U \left(e^{i\alpha} - \frac{c^2 e^{-i\alpha}}{c^2 e^{i2\pi}} \right) - i \frac{k}{2\pi c e^{i\pi}} \\ &= -U (e^{i\alpha} - e^{-i\alpha}) + i \frac{k}{2\pi c} \\ &\quad [ce^{i\pi} = c \cos \pi + i c \sin \pi = -c]. \end{aligned}$$

Remembering that

$$\frac{e^{i\alpha} - e^{-i\alpha}}{2i} = \sin \alpha$$

$$\cancel{k} = 2\pi U_0 \cdot 2\cancel{c} \sin \alpha$$

$$k = 4\pi U_0 \sin \alpha$$

It is possible to obtain an expression which would give the velocity at every point in the ζ plane in the same manner as for the case of non-circulatory motion. The only need of a velocity relation is for plotting a pressure distribution diagram. In this case only the absolute values of the velocity of the air in contact with the line are needed. Because this line is the transform of the circle of radius "c" it is only necessary to transform those velocities on this circle. The value of z for this circle is

$$z = ce^{i\theta}$$

Substituting this value of z and the value of the circulation, k , in the expression for the first derivative of the velocity potential,

$$\begin{aligned} \frac{dw}{dz} &= -U \left(e^{i\alpha} - \frac{c^2 e^{-i\alpha}}{z^2 e^{i2\theta}} \right) - i \frac{2\pi U_0 \sin \alpha}{2\pi z e^{i\theta}} \\ &= -\frac{U}{e^{i\theta}} \left(e^{i(\theta+\alpha)} - e^{-i(\theta+\alpha)} \right) - i \frac{2U \sin \alpha}{e^{i\theta}} \\ &= -\frac{12U}{e^{i\theta}} \left[\sin(\theta + \alpha) + \sin \alpha \right] \end{aligned}$$

Recalling that

$$|z| = \sqrt{(x+iy)(x-iy)}$$

the absolute value of $\frac{dw}{dz}$, which is a complex number,

becomes

$$\left| \frac{dw}{dz} \right| = \sqrt{\left(1 - \frac{2U}{e^{i\theta}} [\sin(\theta + \alpha) + \sin \alpha]\right) \left(\frac{12U}{e^{-i\theta}} [\sin(\theta + \alpha) + \sin \alpha]\right)}$$

$$= 2U [\sin(\theta + \alpha) + \sin \alpha]$$

For

$$z = ce^{i\theta}$$

$$\frac{dz}{d\theta} = (1 - \frac{\theta^2}{\beta^2 e^{i2\theta}}) = \frac{1}{e^{i\theta}} [e^{i\theta} - e^{-i\theta}] = \frac{12 \sin \theta}{e^{i\theta}}$$

$$\left| \frac{dz}{d\theta} \right| = 2 \sin \theta$$

$$\left| \frac{dz}{d\theta} \right| = \frac{1}{2 \sin \theta}$$

But

$$\left| \frac{dw}{d\theta} \right| = \sqrt{(u' - iv')(u' + iv')} = \sqrt{u'^2 + v'^2} = q'$$

$$= \left| \frac{dw}{dz} \right| \cdot \left| \frac{dz}{d\theta} \right|$$

Therefore

$$q' = \frac{2U [\sin(\theta + \alpha) + \sin \alpha]}{\beta \sin \theta}$$

$$= U \left[\frac{\sin \theta \cos \alpha + \cos \theta \sin \alpha + \sin \alpha}{\sin \theta} \right]$$

$$= U \left[\cos \alpha + \sin \alpha \left(\frac{1 + \cos \theta}{\sin \theta} \right) \right]$$

But

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2\cos^2 \theta - 1$$

or

$$2 \cos^2 \theta = 1 + \cos 2\theta$$

or

$$2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta$$

$$\text{Likewise } \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$\text{from which } \frac{1 + \cos \theta}{\sin \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{\beta \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \cot \frac{\theta}{2}$$

Substituting in the expression for q'

$$q' = U \left[\cos \alpha + \sin \alpha \cot \frac{\theta}{2} \right]$$

Substitution of $\theta = 0$ in this equation in order to obtain q' at the leading edge gives

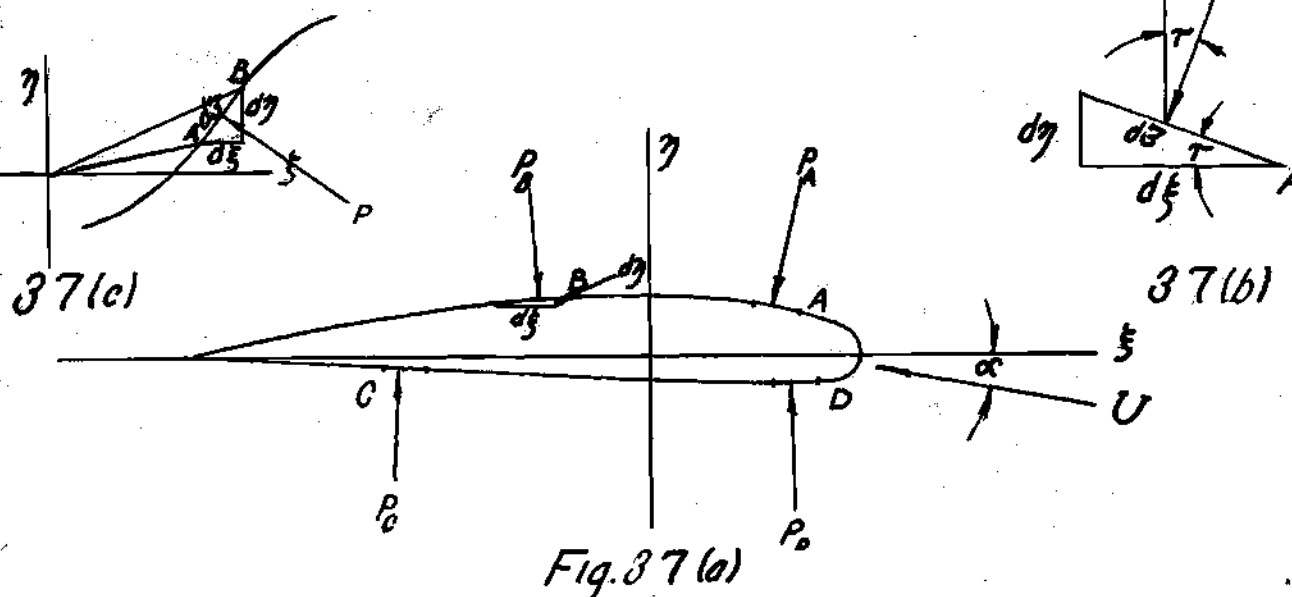
$$q' = \infty.$$

Inspection shows that at every other point, q' is finite.

COMPUTATION OF LIFT OF A FLAT PLATE.

Before beginning the analysis of lift of a flat plate it is necessary to define a few terms and derive a few expressions relating to the lift of an airfoil of finite thickness.

The lift of an airfoil is defined as the component of the resultant air force which is normal to the direction of the relative air. The angle of attack is defined as the angle between the chord, or some other line of reference, of the airfoil and the direction of the velocity of the air at infinity.



Let p be the pressure at any point of an airfoil. The elementary area $[dA] \times 1 \text{ ft.}]$ which p acts upon is shown as a line in Fig. 37 (b).

Total force $p|d\mathcal{Z}|$

Component parallel to $\eta[\perp \text{ to } \xi] = p|d\mathcal{Z}|\cos\tau = dF_\eta$

Component parallel to $\xi = -p|d\mathcal{Z}|\sin\tau = dF_\xi$

Inspection of Fig. 37(b) shows that

$$|d\mathcal{Z}|\cos\tau = d\xi$$

$$|d\mathcal{Z}|\sin\tau = d\eta$$

so that

$$dF_\eta = p d\xi$$

$$dF_\xi = -p d\eta$$

To understand the signs, the portion of the airfoil, $|d\mathcal{Z}|$, should be as in Fig. 37(c). In that case $d\xi$, $d\eta$ and τ are all positive. The pressure is always positive for negative pressure is impossible in air. Hence for Fig. 37(c),

(A positive pressure)(a positive $d\xi$) = a positive dF_η .

(A positive pressure)(a positive $d\eta$) = a negative dF_ξ .

In Fig. 37(b)

(A positive pressure)(a negative $d\xi$) = a negative dF_η

(A positive pressure)(a positive $d\eta$) = a negative dF_ξ

When the expression is written the sign used for the entire expression must combine with the sign of the independent variable at every point so as to give the correct sign to the component of the force. Thus at point B in Fig. 37(a), $d\eta$ is negative and this combines with the negative sign in front to give a positive dF_ξ . At point C, $d\xi$ is positive which makes dF_η positive. At point D $d\eta$ is negative which makes dF_ξ positive.

It is evident that the summation of all the elementary components of either force will give all of that force. This is accomplished by taking the line integral all around the surface of the airfoil. So that

$$F_{\eta} = \int_{\text{curve}} p d\xi$$

$$F_{\xi} = - \int_{\text{curve}} p d\eta$$

We can obtain the expressions for $d\xi$ and $d\eta$ for the flat plate from the transformation function.

$$\begin{aligned} \zeta = \xi + i\eta = z + \frac{c^2}{z} &= x + iy + \frac{c^2}{x + iy} = x + iy + \frac{c^2(x - iy)}{(x + iy)(x - iy)} \\ &= x + \frac{cx}{x^2 + y^2} + i \left[y - \frac{c^2 y}{x^2 + y^2} \right] \end{aligned}$$

Equating the reals and the imaginaries

$$\xi = x \left[1 + \frac{c^2}{x^2 + y^2} \right]$$

$$\eta = y \left[1 - \frac{c^2}{x^2 + y^2} \right]$$

For points on the circle of radius "c" which transforms to the line [the flat plate].

$$x^2 + y^2 = c^2$$

So for points on the line

$$\xi = 2x = 2c \cos \theta$$

$$\eta = 0$$

and

$$d\xi = -2c \sin \theta d\theta.$$

We must use Bernoulli's theorem in order to obtain an expression for p .

$$\text{At infinity} \begin{cases} \text{pressure} = p_0 \\ \text{velocity} = U \end{cases}$$

$$p_0 + \frac{1}{2} \rho U^2 = H$$

$$\text{At any point on the line} \begin{cases} \text{pressure} = p \\ \text{velocity} = q' = U [\cos \alpha + \sin \alpha \cot \frac{\theta}{2}] \end{cases}$$

$$p + \frac{1}{2} \rho q'^2 = H$$

$$p + \frac{1}{2} \rho U^2 [\cos^2 \alpha + 2 \sin \alpha \cos \alpha \cot \frac{\theta}{2} + \sin^2 \alpha \cot^2 \frac{\theta}{2}] = H$$

This Equation is not assumed to hold where $q' = \infty$.

We may now write

$$p + \frac{1}{2} \rho U^2 [\cos^2 \alpha + 2 \sin \alpha \cos \alpha \cot \frac{\theta}{2} + \sin^2 \alpha \cot^2 \frac{\theta}{2}] = p_0 + \frac{1}{2} \rho U^2$$

Solving for p ,

$$p = p_0 + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho U^2 [\cos^2 \alpha + 2 \sin \alpha \cos \alpha \cot \frac{\theta}{2} + \sin^2 \alpha \cot^2 \frac{\theta}{2}]$$

Substituting for $d\mathbf{s}$ and p ,

$$F_{\mathcal{J}} = \int_{\epsilon}^{2\pi-\epsilon} \left\{ p_0 + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho U^2 [\cos^2 \alpha + 2 \sin \alpha \cos \alpha \cot \frac{\theta}{2} + \sin^2 \alpha \cot^2 \frac{\theta}{2}] \right\} [-2c \sin \theta] d\theta$$

We have to take the limits as shown because of the infinite velocity at $\theta = 0$ and 2π . After integration and substitution of limits, ϵ will be allowed to approach zero and if determinate $F_{\mathcal{J}}$ will be the result where $\epsilon = 0$.

$F_{\mathcal{J}}$ can be divided into two parts, $F_{\mathcal{J}}'$ and $F_{\mathcal{J}}''$

Let

$$\begin{aligned} F_{\mathcal{J}}' &= 2c \left\{ p_0 + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho U^2 \cos^2 \alpha \right\} \int_0^{2\pi} -\sin \theta d\theta \\ &= 2c \left\{ p_0 + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho U^2 \cos^2 \alpha \right\} \cos \theta \Big|_0^{2\pi} \\ &= 2c \left\{ p_0 + \frac{1}{2} \rho U^2 - \frac{1}{2} \rho U^2 \cos^2 \alpha \right\} [1 - 1] = 0 \end{aligned}$$

Let

$$F''_{\eta} = F_{\eta} = \int_{\epsilon}^{2\pi-\epsilon} \frac{1}{2} \rho U^2 [2 \sin \alpha \cos \alpha \cot \frac{\theta}{2} + \sin^2 \alpha \cot^2 \frac{\theta}{2}] [-2c \sin \theta d\theta]$$

The evaluation of this depends on

$$\begin{aligned} & \begin{cases} \int \cot \frac{\theta}{2} \sin \theta d\theta \\ \int \cot^2 \frac{\theta}{2} \sin \theta d\theta \end{cases} \\ & \cot \frac{\theta}{2} = \frac{1 + \cos \theta}{\sin \theta} \\ & \int \cot \frac{\theta}{2} \sin \theta d\theta = \int \frac{(1 + \cos \theta) \sin \theta d\theta}{\sin \theta} = \theta + \sin \theta \\ & \int \cot^2 \frac{\theta}{2} \sin \theta d\theta = \int (1 + \csc^2 \frac{\theta}{2}) \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta \\ & = \int 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta + 2 \int \frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta}{\sin^2 \frac{\theta}{2}} \\ & = 2 \sin^2 \frac{\theta}{2} + 4 \int \cot \frac{\theta}{2} \cdot \frac{d\theta}{2} \end{aligned}$$

$$\int \cot \frac{\theta}{2} \cdot \frac{d\theta}{2} = \log \sin \frac{\theta}{2}$$

On substituting

$$F_{\eta} = 2 \rho c U^2 \sin \alpha \cos \alpha [\theta + \sin \theta]_0^{2\pi} + \rho c U^2 \sin^2 \alpha [2 \sin^2 \frac{\theta}{2}]_0^{2\pi} + \rho c U^2 \sin^2 \alpha [4 \log \sin \frac{\theta}{2}]_{\epsilon}^{2\pi-\epsilon}$$

$$[\sin \theta]_0^{2\pi} = 0 = [2 \sin^2 \frac{\theta}{2}]_0^{2\pi}$$

$$[4 \log \sin \frac{\theta}{2}]_{\epsilon}^{2\pi-\epsilon} = 4 \log \sin(\pi - \frac{\epsilon}{2}) - 4 \log \sin \frac{\epsilon}{2} = 4 \log \sin \frac{\epsilon}{2} - 4 \log \sin \frac{\epsilon}{2} = 4 \log \frac{\sin \frac{\epsilon}{2}}{\sin \frac{\epsilon}{2}}$$

$$\text{Evidently } \lim_{\epsilon \rightarrow 0} \log \frac{\sin \frac{\epsilon}{2}}{\sin \frac{\epsilon}{2}} = \log 1 = 0$$

Therefore,

$$F_{\eta} = 4 \pi \rho c U^2 \sin \alpha \cos \alpha$$

$$F_p = - \int p d\eta$$

Since $\eta = 0$ at all points of a flat plate, $d\eta = 0$ and $dF_z = 0$ at every point except at $\theta = 0$ or 2π .

At $\theta = 0$.

$$dF_z = 0.00 \text{ [indeterminate]}.$$

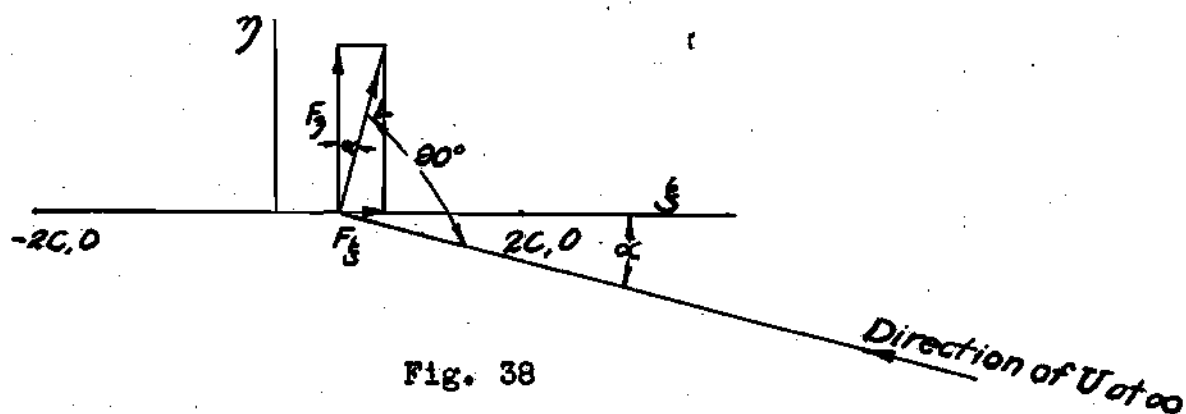


Fig. 38

Because the lift is perpendicular to the direction of U at ∞ , the angle between L and F_z is α .

$$L \cos \alpha = F_z = 4\pi \rho c U^2 \sin \alpha \cos \alpha$$

$$L = 4\pi \rho c U^2 \sin \alpha$$

$$F_z = 4\pi \rho c U^2 \sin^2 \alpha.$$

Thus for this case we have evaluated 0.00 .

The expression for the lift may be written,

$$L = \rho U \cdot [4\pi c U \sin \alpha] = \rho U k.$$

This proves that the lift of the flat plate, in theory at least, is identical to the lift of the cylinder. The area of the flat plate is $4c \times 1$ ft. By definition,

$$L = \rho k_1 S U^2 = \rho k_1 [4c] U^2 = \rho [4c] U^2 \pi \sin \alpha$$

or

$$L = \frac{1}{2} \rho C_L S U^2 = \frac{1}{2} \rho C_L [4c] U^2 = \rho [4c] U^2 \pi \sin \alpha$$

From these relations

$$k_L = \pi \sin \alpha \quad [\text{British}]$$

$$C_L = 2\pi \sin \alpha \quad [\text{N.A.C.A.}]$$

For small angles,

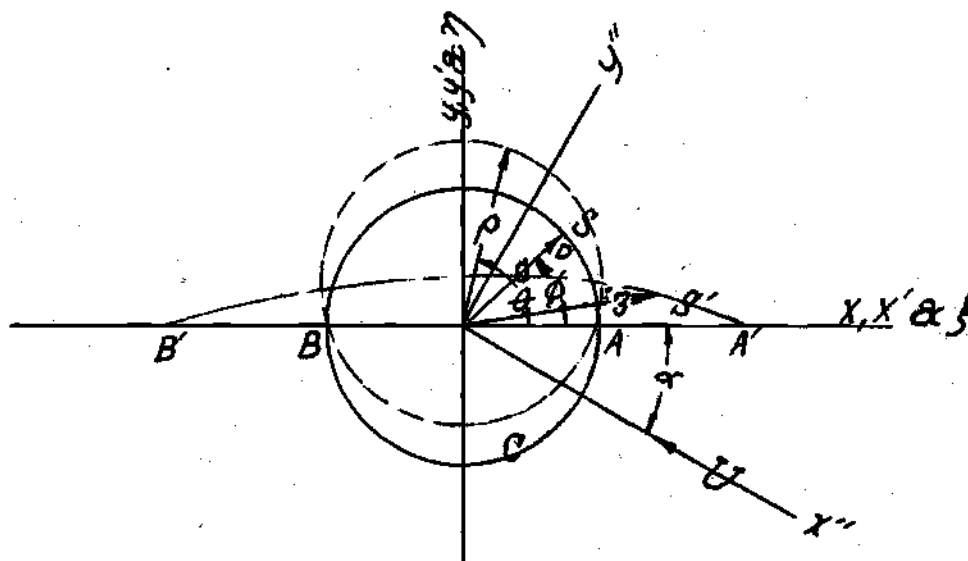
$$\sin \alpha = \alpha$$

$$k_L = \pi \alpha$$

$$C_L = 2\pi \alpha$$

Experiment agrees with this result. The lift coefficient plotted against the angle of attack is a straight line for small angles, beginning with $\alpha = 0$. This theory makes the slope of the lift coefficient against angle of attack π or 2π , according to the system, for an infinite aspect ratio. Glauert says that in the British system for actual airfoils it is nearer 3.00 than 3.1416. The true values lie approximately in the range from 2.6 to 2.9. Experiment and theory agree that for a finite aspect ratio of six of a rectangular airfoil the slope is near 2.23 in the British system. These are the approximately correct figures for α plotted in radians. Slight variations with the shape of the profile contour are found. In the British system the lift coefficient at small angles of attack increases 0.055 per degree for an infinite aspect ratio and 0.039 per degree for an aspect ratio of six. As defective as a flat plate is for production of lift its lift coefficient at very small angles of attack increases nearly 0.037 per degree for an aspect ratio of six.

GLAUERT'S ANALYSIS OF THE LIFT OF A THIN AIRFOIL OF ARBITRARY SHAPE.



The circle of radius "a" transforms into the straight line A'B' extending from $\zeta = -2a$ to $\zeta = +2a$ by use of

$$\zeta = z + \frac{a^2}{z} \quad (1)$$

S' is the mean of the upper and lower surfaces of a thin airfoil and the primary assumption is that it does not deviate much from the straight line A'B'.

S is a curve which differs but little from a circle. It is called a near circle.

For mathematical simplification our transformation is made in two steps:

- a. S' is transformed into S .
- b. S is transformed into C .

"Transformation" means in this case that a mathematical relation is established between ρ & ζ and between "a" & ρ .

Equation (1) is not used in this analysis. It is merely introduced to show that since by use of it the circle transforms into

a straight line that a curve S' , from A' to B' which is almost a straight line, must transform to a closed curve S which is almost a circle by use of

$$z = z' + \frac{a^2}{z'} \quad (1a)$$

Where $z' =$ complex co-ordinate of near circle.

We may also use $z' = \rho e^{i\theta}$ [ρ is a variable].

Remembering that A on the circle transformed to A' by use of equation (1) we see that A' on S' must transform by use of (1a) back to the same point A . But since we used (1a) for this latter transformation A must now be a point on the near circle S .

If this isn't sufficiently clear then the following mathematical analysis will clear the point.

The point A' on S' has the co-ordinates

$$z = \xi + i\eta = 2a$$

which gives

$$\begin{cases} \xi = 2a \\ \eta = 0 \end{cases}$$

Likewise the co-ordinates of B' are

$$\begin{cases} \xi = -2a \\ \eta = 0 \end{cases}$$

Hence the co-ordinates of the point on the near circle which is the transform of point A' on S' are obtained from

$$\begin{aligned} z = 2a + i0 &= \rho e^{i\theta} + \frac{a^2}{\rho e^{i\theta}} = \rho e^{i\theta} + \frac{a^2}{\rho} e^{-i\theta} \\ &= \rho(\cos\theta + i\sin\theta) + \frac{a^2}{\rho}(\cos\theta - i\sin\theta) \\ &= \left(\rho + \frac{a^2}{\rho}\right) \cos\theta + i\left(\rho - \frac{a^2}{\rho}\right) \sin\theta \end{aligned}$$

Equating reals

$$2a = \left(\rho + \frac{a^2}{\rho}\right) \cos\theta$$

Equating imaginaries

$$0 = \left(\rho - \frac{a^2}{\rho}\right) \sin \theta$$

Assume that $\sin \theta$ is not equal to zero, then

$$\frac{\rho^2 - a^2}{\rho} = 0$$

$$\text{and } \rho = a$$

This substituted in the equation containing $\cos \theta$ gives

$$2a = \left(a + \frac{a^2}{a}\right) \cos \theta = 2a \cos \theta$$

$$\text{or } \cos \theta = 1$$

$$\theta = 0$$

$$\text{and } \sin \theta = 0$$

If we had assumed that

$$\sin \theta = 0$$

then we would have obtained

$$2a = \left(\rho + \frac{a^2}{\rho}\right)$$

$$\text{and } \rho^2 - 2a\rho + a^2 = 0$$

giving

$$\rho = \frac{2a \pm \sqrt{4a^2 - 4a^2}}{2} = a.$$

Therefore A' transforms to a point on the axis of reals

$$x' = a$$

$$y' = 0.$$

Hence this puts point A on the near circle.

Likewise B' transforms to point B.

This puts two points of the near circle on the axis of reals at $\pm a$ and therefore S and C must intersect at these points.

After all points are located on S by transforming the known

points on S' another and different transformation must be made from the near circle S to the true circle C . It is possible to transform S' directly back to C by use of a transformation formula different from (1a) but the mathematical analysis seems to be easier if done in two steps: first, from S' to S ; second, from S to C . This second transformation will transform points A and B from the near circle to altogether different points on the circle C . In other words the angle ϕ for the point on the circle which is the transform of the point A on the near circle will not be zero degrees.

Because S' is sharp [has a cusp] at A' and at B' , these points are singular points and the velocity at A' will be infinite for every angle of attack except the one for which the point on the circle C , which transforms to A on the near circle which in turn transforms to A' , is a stagnation point. By Joukowski's hypothesis the circulation will be determined so that the flow will leave the trailing edge smoothly with a finite velocity; so for this reason the velocity at B' will not be infinite. For the same reason that A on S' does not transform to A on C , a point on the circle such as D will not transform to the point on the near circle which lies on the extension of the radius to D . The z co-ordinate of point D will make a different angle with the axis of reals from the z' co-ordinate of the point on S which transforms to point D . This will be more evident as the analysis proceeds.

Let

$$\frac{|\rho - a|}{|a|} = |r| \quad [\text{a ratio which varies with } \rho \text{ and, therefore, with } \theta]$$

then

$$|\rho| = |a| + |r| \cdot |a|$$

and

$$z' = \rho e^{i\theta} = ae^{i\theta} + are^{i\theta} = a(1+r)e^{i\theta}.$$

Hence S will transform to S' or vice-versa by

$$\begin{aligned} \mathcal{Z} = z' + \frac{a^2}{z'} &= a(1+r)e^{i\theta} + \frac{a^2}{a(1+r)e^{i\theta}} = a(1+r)e^{i\theta} + \frac{a}{1+r}e^{-i\theta} \quad (2) \\ &= a(1+r) [\cos\theta + i \sin\theta] + \left(\frac{a}{1+r}\right) [\cos\theta - i \sin\theta] \\ \mathcal{Z} = \xi + i\eta &= a \left[(1+r) + \frac{1}{(1+r)} \right] \cos\theta + ia \left[(1+r) - \frac{1}{(1+r)} \right] \sin\theta \end{aligned}$$

Equating reals

$$\xi = a \left[(1+r) + \frac{1}{1+r} \right] \cos\theta$$

Equating imaginaries

$$\eta = a \left[(1+r) - \frac{1}{1+r} \right] \sin\theta$$

Placing the leading edge and the trailing edge of the mean of the upper and lower surfaces of a R.A.F. 15 airfoil on the axis of reals, the value of η where $\xi = 0$ [50% of chord] is 0.0221 x chord. The chord will be 4a. For this point,

$$\begin{aligned} \xi = 0 &= a \left[(1+r) + \frac{1}{(1+r)} \right] \cos\theta \\ a \left[(1+r) + \frac{1}{(1+r)} \right] &\text{ can not be zero, so this makes} \end{aligned}$$

$$\cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

$$\eta = .0221 \times 4a = .0884a = a \left[(1+r) - \frac{1}{(1+r)} \right] \sin \frac{\pi}{2}$$

$$1+r - \frac{1}{1+r} = .0884$$

$$(1+r)^2 - 1 = .0884(1+r)$$

$$1 + 2r + r^2 - 1 = .0884 + .0884r$$

$$r^2 + (2 - .0884)r - .0884 = 0$$

$$r = .0452$$

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \dots$$

$$\begin{aligned} \frac{1}{1+.0452} &= 1 - .0452 + .0020 - .00009 + \dots \\ &= .9548 + .0020 \end{aligned}$$

The error in discarding r^3 is about 2% and therefore negligible. But this is one of the large values of r , if not the largest, and for most of the other values the error is much less.

Taking

$$\frac{1}{1+r} = 1 - r$$

$$\xi = a(1+r+1-r) \cos \theta = 2a \cos \theta$$

$$\eta = a(1+r-1+r) \sin \theta = 2ar \sin \theta$$

These are the relations for transferring from the airfoil to the near circle.

For points on the circle "c"

$$z = ae^{i\phi}$$

For points on the near circle "s"

$$z' = a(1+r)e^{i\theta}$$

$$\frac{z'}{z} = \frac{(1+r)e^{i\theta}}{e^{i\phi}} = (1+r)e^{i(\theta-\phi)}$$

$$z' = z(1+r)e^{i(\theta-\phi)}$$

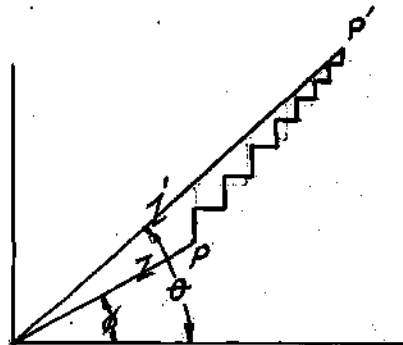
It takes a different transformation to transform from the circle "c" to the near circle "s". Glauert uses

$$z' = z + iA_1 \frac{az}{z} + iA_2 \frac{a^2 z}{z^2} + iA_3 \frac{a^3 z}{z^3} + \dots \quad (5)$$

This is permissible because where

$$z = \infty$$

$$z' = z.$$



This figure is a crude picture of the transformation of point P to P'.

(5) is the expansion of

$$\begin{aligned} z' &= z \left(1 + i \sum_{n=1}^{\infty} A_n \frac{a^n}{z^n} \right) = z \left[1 + i \sum_{n=1}^{\infty} \frac{A_n a^n}{z^n e^{in\phi}} \right] \\ &= z \left[1 + i \sum_{n=1}^{\infty} A_n (\cos n\phi - i \sin n\phi) \right] \\ &= z \left[1 + \sum_{n=1}^{\infty} A_n \sin n\phi + i \sum_{n=1}^{\infty} A_n \cos n\phi \right] \quad (6) \end{aligned}$$

Equating (4) and (6)

$$\frac{1}{2}(1+r)e^{i(\theta-\phi)} = \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} A_n \sin n\phi + i \sum_{n=1}^{\infty} A_n \cos n\phi \right)$$

Let $\theta - \phi = \epsilon$

$$e^{i(\theta-\phi)} = e^{i\epsilon} = \cos \epsilon + i \sin \epsilon.$$

Substituting,

$$\begin{aligned} (1+r)(\cos \epsilon + i \sin \epsilon) &= 1 + \sum_{n=1}^{\infty} A_n \sin n\phi + i \sum_{n=1}^{\infty} A_n \cos n\phi \\ \cos \epsilon + r \cos \epsilon + i(\sin \epsilon + r \sin \epsilon) &= 1 + \sum_{n=1}^{\infty} A_n \sin n\phi \\ &\quad + i \sum_{n=1}^{\infty} A_n \cos n\phi \end{aligned}$$

ϵ will be small. It is assumed that

$$\cos \epsilon = 1$$

$$r \sin \epsilon = 0$$

$$\sin \epsilon = \epsilon$$

Substituting,

$$1 + r + i\epsilon = 1 + \sum_1^{\infty} A_n \sin n\phi + i \sum_1^{\infty} A_n \cos n\phi$$

Equating reals,

$$1 + r = 1 + \sum_1^{\infty} A_n \sin n\phi$$

Equating imaginaries,

$$\epsilon = \sum_1^{\infty} A_n \cos n\phi$$

Substituting $(\theta - \epsilon)$ for ϕ

$$r = \sum_1^{\infty} A_n \sin n(\theta - \epsilon) = \sum_1^{\infty} A_n (\sin n\theta \cos n\epsilon - \cos n\theta \sin n\epsilon) = \sum_1^{\infty} A_n \sin n\theta \quad (7)$$

$$\epsilon = \sum_1^{\infty} A_n \cos n(\theta - \epsilon) = \sum_1^{\infty} A_n (\cos n\theta \cos n\epsilon + \sin n\theta \sin n\epsilon) = \sum_1^{\infty} A_n \cos n\theta \quad (7)$$

Note: Suppose $\begin{cases} \phi = 90^\circ \\ \theta = 85^\circ \end{cases}$ and that the constants A_1, A_2, A_3 , etc.

were arbitrarily chosen to be $\frac{4}{10}, \frac{4}{100}, \frac{4}{1000}$, etc. We

know that these constants must decrease rapidly in order for r and ϵ to be small quantities. Then the arbitrary value of r would be,

$$r = 4 \left(\frac{1}{10} \sin \phi + \frac{1}{100} \sin 2\phi + \frac{1}{1000} \sin 3\phi + \frac{1}{10,000} \sin 4\phi + \dots \right)$$

$$(r)_{\phi=90^\circ} = 4 \left(\frac{1}{10} - \frac{1}{1000} + \frac{1}{100,000} - \dots \right) = .3960$$

Furthermore using the approximation

$$\sum_1^{\infty} A_n \sin n\phi = \sum_1^{\infty} A_n \sin n\theta$$

The arbitrary value of r would be,

$$r = 4 \left(\frac{\sin 85^\circ}{10} + \frac{\sin 10^\circ}{100} - \frac{\sin 75^\circ}{1000} - \frac{\sin 20^\circ}{10,000} + \dots \right)$$

$$= 4 \left(\frac{.9962}{10} + \frac{.174}{100} - \frac{.966}{1000} - \frac{.342}{10,000} + \dots \right)$$

$$= 4 \times .10036 = .4 \text{ approximately.}$$

This demonstrates a nice degree of approximation although ϵ is taken as large as 5° and r so large that its square is not negligible.

Expanding the expression for r ,

$$r = A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \dots + A_k \sin k\theta + \dots + A_n \sin n\theta$$

Multiplying both sides by $\sin k\theta \, d\theta$

$$r \sin k\theta \, d\theta = A_1 \sin \theta \sin k\theta \, d\theta + A_2 \sin 2\theta \sin k\theta \, d\theta + \dots + A_k \sin^2 k\theta \, d\theta + \dots$$

Recalling,

$$\cos(m+n)\theta = \cos m\theta \cos n\theta - \sin m\theta \sin n\theta$$

$$\cos(m-n)\theta = \cos m\theta \cos n\theta + \sin m\theta \sin n\theta$$

Subtracting the latter,

$$\sin m\theta \sin n\theta = \frac{1}{2} [\cos(m-n)\theta - \cos(m+n)\theta]$$

Integrating both sides between the limits of 0 and π ,

$$\begin{aligned} \int_0^\pi \sin m\theta \sin n\theta \, d\theta &= \frac{1}{2} \int_0^\pi [\cos(m-n)\theta - \cos(m+n)\theta] \, d\theta \\ &= \frac{1}{2} \left[\frac{\sin(m-n)\theta}{m-n} - \frac{\sin(m+n)\theta}{m+n} \right]_0^\pi \end{aligned}$$

If m is not equal to n , substitution of the limits gives

$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0$$

If m is equal to n , substitution of the limits makes

$$\frac{1}{2} \left[\frac{\sin(m-n)\theta}{m-n} - \frac{\sin(m+n)\theta}{m+n} \right]_0^\pi = \frac{0}{0} \quad [\text{Indeterminate.}]$$

But if m is equal to n

$$\begin{aligned} \int_0^\pi \sin m\theta \sin n\theta \, d\theta &= \frac{1}{2} \int_0^\pi \cos(0) \, d\theta - \frac{1}{2} \int_0^\pi \cos 2m\theta \, d\theta \\ &= \left[\frac{\theta}{2} - \frac{1}{4m} \sin 2m\theta \right]_0^\pi = \frac{\pi}{2} \end{aligned}$$

This proves that

$$\int_0^{\pi} r \sin k\theta \, d\theta = \int_0^{\pi} A_k \sin^2 k\theta \, d\theta = A_k \frac{\pi}{2} \quad (8)$$

Which means that

$$\begin{aligned} A_1 &= \frac{2}{\pi} \int_0^{\pi} r \sin \theta \, d\theta \\ A_2 &= \frac{2}{\pi} \int_0^{\pi} r \sin 2\theta \, d\theta \\ A_3 &= \frac{2}{\pi} \int_0^{\pi} r \sin 3\theta \, d\theta \\ &\text{etc,} \end{aligned}$$

Expanding the expression for \mathcal{E} ,

$$\mathcal{E} = A_1 \cos \theta + A_2 \cos 2\theta + A_3 \cos 3\theta + \text{-----}$$

For $\theta = \pi$

$$\mathcal{E}_0 = -A_1 + A_2 - A_3 + A_4 - \text{-----}$$

or

$$-\mathcal{E}_0 = A_1 - A_2 + A_3 - A_4 + \text{-----}$$

But

$$\theta - \phi = \mathcal{E}$$

and

$$\pi - \phi_0 = -\mathcal{E}_0$$

or

$$\phi_0 = \pi - (-\mathcal{E}_0) = \pi + \mathcal{E}_0$$

ϕ_0 is the angle of the point on the circle which transforms to B which in turn transforms to the trailing edge B'. It would be possible to determine \mathcal{E}_0 by solving for A_1 , then A_2 , then A_3 , etc. according to equation (8) but this would be too laborious. The following analysis provides a better method.

$$\begin{aligned} r &= \sum A_n \sin n\theta \\ &= A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \text{-----} \end{aligned}$$

Substituting this expression for r in the expressions for A_1 , A_2 , A_3 , etc.

$$\begin{aligned}
 A_1 &= \frac{2}{\pi} \int_0^\pi (A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \dots) \sin \theta \, d\theta \\
 A_2 &= \frac{2}{\pi} \int_0^\pi (A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \dots) \sin 2\theta \, d\theta \\
 A_3 &= \frac{2}{\pi} \int_0^\pi (A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \dots) \sin 3\theta \, d\theta \\
 &\text{etc.}
 \end{aligned}$$

Substituting these values in the expression for \mathcal{E}_0 ,

$$\begin{aligned}
 -\mathcal{E}_0 &= A_1 - A_2 + A_3 - A_4 + \dots \\
 &= \frac{2}{\pi} \int_0^\pi [A_1 \sin \theta + A_2 \sin 2\theta + \dots] \sin \theta \, d\theta - [(A_1 \sin \theta + A_2 \sin 2\theta + \dots) \sin 2\theta \, d\theta] + \dots \\
 &= \frac{2}{\pi} \int_0^\pi [A_1 \sin \theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \dots] [\sin \theta - \sin 2\theta + \sin 3\theta - \dots] \, d\theta \\
 &= \frac{2}{\pi} \int_0^\pi r [\sin \theta - \sin 2\theta + \sin 3\theta - \sin 4\theta + \dots] \, d\theta
 \end{aligned}$$

We can obtain a function to replace this series in the following manner:

$$\sin \frac{\theta}{2} = \frac{e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}}{2j}$$

$$\cos \frac{\theta}{2} = \frac{e^{j\frac{\theta}{2}} + e^{-j\frac{\theta}{2}}}{2}$$

$$\begin{aligned}
 \tan \frac{\theta}{2} &= \frac{1}{j} \frac{e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}} + e^{-j\frac{\theta}{2}}} = \frac{1}{j} \frac{e^{j\frac{\theta}{2}} \cdot e^{j\frac{\theta}{2}} - e^{j\frac{\theta}{2}} \cdot e^{-j\frac{\theta}{2}}}{e^{j\frac{\theta}{2}} \cdot e^{j\frac{\theta}{2}} + e^{j\frac{\theta}{2}} \cdot e^{-j\frac{\theta}{2}}} = \frac{1}{j} \frac{e^{j\theta} - 1}{e^{j\theta} + 1} \\
 &= \frac{1}{j} \frac{e^{j\theta}}{1 + e^{j\theta}} - \frac{1}{j} \frac{1}{1 + e^{j\theta}}
 \end{aligned}$$

By dividing numerator by denominator,

$$\frac{e^{j\theta}}{1 + e^{j\theta}} = \frac{e^{j\theta} - e^{j2\theta} + e^{j3\theta} - e^{j4\theta} + e^{j5\theta} - \dots}{1 + e^{j\theta}}$$

$$\frac{1}{1 + e^{j\theta}} = 1 - \frac{e^{j\theta} + e^{j2\theta} + e^{j3\theta} + e^{j4\theta} + e^{j5\theta} + \dots}{1 + e^{j\theta}}$$

By subtraction,

$$\frac{1}{j} \frac{e^{j\theta}}{1 + e^{j\theta}} - \frac{1}{j} \frac{1}{1 + e^{j\theta}} = \frac{1}{j} \left[-1 + 2e^{j\theta} - 2e^{j2\theta} + 2e^{j3\theta} - 2e^{j4\theta} + \dots \right]$$

Substituting,

$$\cos \theta + j \sin \theta = e^{j\theta}$$

$$\cos 2\theta + j \sin 2\theta = e^{j2\theta}$$

$$\cos 3\theta + i \sin 3\theta = e^{i3\theta}$$

etc.

$$0 + i \tan \frac{\theta}{2} = \left[-1 + 2 \cos \theta + i 2 \sin \theta - 2 \cos 2\theta - i 2 \sin 2\theta + 2 \cos 3\theta + i 2 \sin 3\theta - \dots \right]$$

Equating reals,

$$0 = -1 + 2 \left[\cos \theta - \cos 2\theta + \cos 3\theta - \cos 4\theta + \dots \right]$$

Equating imaginaries,

$$\tan \frac{\theta}{2} = 2 \left[\sin \theta - \sin 2\theta + \sin 3\theta - \sin 4\theta + \dots \right]$$

Therefore,

$$-\epsilon_0 = \frac{2}{\pi} \int_0^\pi r \tan \frac{\theta}{2} d\theta$$

But $\eta = 2a \sin \theta$

$$\tan \frac{\theta}{2} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \frac{\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{\cos \frac{\theta}{2} \cos \frac{\theta}{2}} = \frac{\frac{\sin \theta}{2}}{\frac{1 + \cos \theta}{2}} = \frac{\sin \theta}{1 + \cos \theta}$$

$$-\epsilon_0 = \frac{1}{\pi} \int_0^\pi \frac{\eta}{2a \sin \theta} \cdot \frac{\sin \theta}{1 + \cos \theta} d\theta = \frac{1}{2\pi a} \int_0^\pi \frac{\eta d\theta}{1 + \cos \theta} \quad (9)$$

We may write then that

$$\phi = \theta - \epsilon$$

$$\text{And } \phi_s = \pi + \frac{1}{2\pi a} \int_0^\pi \frac{\eta d\theta}{1 + \cos \theta}$$

Where ϕ_s is the angle for the stagnation point on circle "c".

TRANSFORMATION OF THE POTENTIAL FUNCTION.

Let $-U$ be the velocity at infinity at the angle α with the x axis. Let x'' be parallel to U . The potential function in the plane of z'' is,

$$w = -U \left(z'' + \frac{a^2}{z''} \right) - \frac{ik}{2\pi} \log \frac{z''}{a}$$

Relation between z and z'' .

$$z'' = ae^{i(\phi + \alpha)} = ae^{i\phi} e^{i\alpha}$$

$$z = ae^{i\phi}$$

$$\frac{z''}{z} = e^{i\alpha}$$

$$z'' = ze^{i\alpha}$$

Substituting $ze^{i\alpha}$ for z'' in w , the potential function of the flow in the z plane is,

$$w = -U\left(ze^{i\alpha} + \frac{a^2}{ze^{i\alpha}}\right) - \frac{ik}{2\pi} \log \frac{ze^{i\alpha}}{a} \quad (10)$$

$$\frac{dw}{dz} = -U\left(e^{i\alpha} - \frac{a^2}{z^2 e^{i\alpha}}\right) - \frac{ik}{2\pi} \cdot \frac{1}{z}$$

At the surface of the circle

$$\begin{aligned} \frac{dw}{dz} &= -U\left(e^{i\alpha} - \frac{a^2}{a^2 e^{i\phi} e^{i\alpha}}\right) - \frac{ik}{2\pi} \cdot \frac{1}{ae^{i\phi}} \\ &= -U\left[\frac{e^{i\alpha} \cdot e^{i\phi}}{e^{i\phi}} - \frac{e^{-i\phi} e^{-i\alpha}}{e^{i\phi}}\right] - \frac{ik}{2\pi} \cdot \frac{1}{ae^{i\phi}} \\ &= -\frac{U}{e^{i\phi}} \left[e^{i(\phi + \alpha)} - e^{-i(\phi + \alpha)} \right] - \frac{ik}{2\pi} \frac{1}{ae^{i\phi}} \\ &= -\frac{U}{e^{i\phi}} \cdot 2i \sin(\phi + \alpha) - \frac{ik}{2\pi} \frac{1}{ae^{i\phi}} \end{aligned}$$

At the stagnation point

$$\begin{aligned} \phi &= \phi_s = \pi + \epsilon_0 \\ \frac{dw}{dz} &= 0 \end{aligned}$$

Solving for the circulation,

$$k = -4\pi aU \sin(\alpha + \pi + \epsilon_0) = 4\pi aU \sin(\alpha + \epsilon_0) \quad (12)$$

The lift of the airfoil per foot of span is

$$L = \rho U k = 4\pi a \rho U^2 \sin(\alpha + \epsilon_0).$$

The lift is therefore zero where $\alpha = -\epsilon_0$. ϵ_0 is the no lift angle of the airfoil and its determination is important. ϵ_0 is determined by graphical integration. For an illustration of its

determination for a R.A.F. 15 airfoil see R.&M. 910 of the Aeronautical Research Committee for 1924-25.

The chord of the airfoil is $4a$. In addition the greatest permissible value of α is so small that,

$$\sin(\alpha + \epsilon_0) = \alpha + \epsilon_0$$

Hence,

$$k_L = \frac{L}{\rho \cdot 4a \cdot U^2} = \frac{4\pi a \rho U^2 (\alpha + \epsilon_0)}{4a \rho U^2} = \pi(\alpha + \epsilon_0) \quad (13)$$

[Note. α and ϵ_0 are to be measured in radians.]

As soon as ϵ_0 has been computed the straight line part of the curve of lift coefficient against angle of attack can be drawn.

VELOCITIES AT THE SURFACE OF THE AIRFOIL.

If a theoretical pressure distribution diagram were desired in order to check an experimental one it would be necessary to know the velocity at every point of the airfoil surface.

Substituting $(\theta - \epsilon)$ for ϕ in the first derivative of the potential function with respect to z ,

$$\begin{aligned} \frac{dw}{dz} &= -\frac{U}{e^{i(\theta-\epsilon)}} \left[2i \sin(\alpha + \theta - \epsilon) \right] - \frac{i 4\pi a U \sin(\alpha + \epsilon_0)}{2\pi a e^{i(\theta-\epsilon)}} \\ &= -\frac{12U}{e^{i(\theta-\epsilon)}} \left[\sin(\alpha + \theta - \epsilon) + \sin(\alpha + \epsilon_0) \right] \end{aligned}$$

Recalling that

$$\begin{aligned} |z| &= \sqrt{(x+iy)(x-iy)} = \sqrt{re^{i\theta} \cdot re^{-i\theta}} \\ \left| \frac{dw}{dz} \right|^2 &= |q|^2 = \left\{ -\frac{12U}{e^{i(\theta-\epsilon)}} \left[\sin(\alpha + \theta - \epsilon) + \sin(\alpha + \epsilon_0) \right] \right. \\ &\quad \left. \left\{ +\frac{12U}{e^{-i(\theta-\epsilon)}} \left[\sin(\alpha + \theta - \epsilon) + \sin(\alpha + \epsilon_0) \right] \right\} \right\} \\ |q| &= 2U \left[\sin(\alpha + \theta - \epsilon) + \sin(\alpha + \epsilon_0) \right] \end{aligned}$$

We indicate the method for determining the absolute value

of the velocity at any point on the surface of the airfoil
by

$$|q_s| = |q_c| \cdot \left| \frac{dz}{dz'} \right| \cdot \left| \frac{dz'}{d\theta} \right|$$

where $|q_c|$ = Absolute value of the velocity at any point on
the circle.

$$\left| \frac{dz}{dz'} \right| = \text{The value of this expression where } z = ae^{i\phi}$$

$$\left| \frac{dz'}{d\theta} \right| = \text{The value of this expression where } z' = a(1+r)e^{i\theta}$$

Now

$$z = ae^{i\phi}$$

$$\frac{dz}{d\phi} = iae^{i\phi}$$

$$\left| \frac{dz}{d\phi} \right| = a$$

Also

$$\begin{aligned} z' &= z \left(1 + i \sum_{n=1}^{\infty} A_n \frac{a^n}{z^n} \right) = ae^{i\phi} \left[1 + i \sum_{n=1}^{\infty} \frac{A_n a^n}{ae^{in\phi}} \right] \\ &= ae^{i\phi} \left[1 + i \sum_{n=1}^{\infty} A_n e^{-in\phi} \right] \\ &= ae^{i\phi} \left[1 + i \sum_{n=1}^{\infty} A_n \cos n\phi + \sum_{n=1}^{\infty} A_n \sin n\phi \right] \end{aligned}$$

But

$$\begin{aligned} r &= \sum_{n=1}^{\infty} A_n \sin n\phi \\ \epsilon &= \sum_{n=1}^{\infty} A_n \cos n\phi \end{aligned}$$

so

$$\begin{aligned} z' &= ae^{i\phi} \left[1 + r + i\epsilon \right] \\ \frac{dz'}{d\phi} &= ae^{i\phi} \left(\frac{dr}{d\phi} + i \frac{d\epsilon}{d\phi} \right) + \left[1 + r + i\epsilon \right] iae^{i\phi} \\ &= ae^{i\phi} \left[\left(\frac{dr}{d\phi} - \epsilon \right) + i \left(1 + r + \frac{d\epsilon}{d\phi} \right) \right] \end{aligned}$$

$$\left| \frac{dz'}{d\phi} \right| = \sqrt{ae^{i\phi} \left[\left(\frac{dr}{d\phi} - \epsilon \right) + i \left(1 + r + \frac{d\epsilon}{d\phi} \right) \right] \cdot ae^{-i\phi} \left[\left(\frac{dr}{d\phi} - \epsilon \right) - i \left(1 + r + \frac{d\epsilon}{d\phi} \right) \right]}$$

$$= a \sqrt{\left(\frac{dr}{d\phi} - \epsilon\right)^2 + \left(1 + r + \frac{d\epsilon}{d\phi}\right)^2}$$

But

$$\left(\frac{dr}{d\phi} - \epsilon\right)^2 = \left(\frac{dr}{d\phi}\right)^2 - 2\epsilon \frac{dr}{d\phi} + \epsilon^2 = 0$$

because each term is of higher order than the precision of this analysis.

Hence

$$\left|\frac{dz'}{d\phi}\right| = a\left(1 + r + \frac{d\epsilon}{d\phi}\right)$$

and because

$$\left|\frac{dz}{dz'}\right| = \left|\frac{dz}{d\phi}\right| \cdot \left|\frac{d\phi}{dz'}\right|$$

$$\left|\frac{dz}{dz'}\right| = \frac{a}{a\left(1 + r + \frac{d\epsilon}{d\phi}\right)} = \frac{1}{1 + r + \frac{d\epsilon}{d\phi}}$$

For the transformation from the near circle to the airfoil,

$$\zeta = z' + \frac{a^2}{z'}$$

At the surface of S'

$$\zeta = a(1+r)e^{i\theta} + a(1-r)e^{-i\theta}$$

$$= 2a \cos \theta + i 2ar \sin \theta$$

$$= 2a(\cos \theta + i r \sin \theta)$$

$$\frac{d\zeta}{d\theta} = 2a \left[-\sin \theta + i(r \cos \theta + \sin \theta \frac{dr}{d\theta}) \right]$$

$$\begin{aligned} \left|\frac{d\zeta}{d\theta}\right| &= \sqrt{2a \left[-\sin \theta + i(r \cos \theta + \frac{dr}{d\theta} \sin \theta) \right] \cdot 2a \left[-\sin \theta - i(r \cos \theta + \frac{dr}{d\theta} \sin \theta) \right]} \\ &= 2a \sqrt{\sin^2 \theta + (r \cos \theta + \frac{dr}{d\theta} \sin \theta)^2} \end{aligned}$$

But

$$\left[r \cos \theta + \frac{dr}{d\theta} \sin \theta \right]^2 = r^2 \cos^2 \theta + 2r \frac{dr}{d\theta} \sin \theta \cos \theta + \left(\frac{dr}{d\theta}\right)^2 \sin^2 \theta \quad \left[\text{This} \right.$$

is taken to be zero because each term contains either the square of an infinitesimal or the product of two infinitesimals.]

So

$$\left| \frac{dz}{d\theta} \right| = 2a \sin \theta$$

Also

$$z' = a(1+r)e^{i\theta}$$

$$\frac{dz'}{d\theta} = a(1+r)i e^{i\theta} + a e^{i\theta} \cdot \frac{dr}{d\theta} = a e^{i\theta} \left[\frac{dr}{d\theta} + i(1+r) \right]$$

because $\left(\frac{dr}{d\theta} \right)^2$ is taken to be zero.

$$\left| \frac{dz'}{d\theta} \right| = a(1+r)$$

$$\left| \frac{dz'}{dz} \right| = \left| \frac{dz'}{d\theta} \right| \cdot \left| \frac{d\theta}{dz} \right|$$

$$= a(1+r) \cdot \frac{1}{2a \sin \theta} = \frac{1+r}{2 \sin \theta}$$

Therefore, making the substitutions for $\left| \frac{dz}{dz'} \right|$ and $\left| \frac{dz'}{dz} \right|$

$$\left| q_{s'} \right| = \left| q_c \right| \cdot \frac{1}{1+r+\frac{d\epsilon}{d\phi}} \cdot \frac{1+r}{2 \sin \theta}$$

We must get everything in terms of θ .

$$\epsilon = \theta - \phi$$

$$-\epsilon = \phi - \theta$$

so

$$\frac{d\epsilon}{d\theta} = 1 - \frac{d\phi}{d\theta}$$

$$-\frac{d\epsilon}{d\phi} = 1 - \frac{d\theta}{d\phi}$$

but $\frac{d\phi}{d\theta} = \frac{d\theta}{d\phi}$ [Almost exactly]

As ϵ is small the curves of ϕ against θ and of θ against ϕ will both have a slope at any point nearly equal to ± 1 .

$$\frac{d\epsilon}{d\phi} = -\frac{d\theta}{d\phi}$$

Making this substitution and the one for $|q_c|$

$$|q_{g'}| = 2U \left[\sin(\alpha + \theta - \varepsilon) + \sin(\alpha + \varepsilon_0) \right] \cdot \left[\frac{1}{1+r-\frac{d\varepsilon}{d\theta}} \cdot \frac{1+r}{2 \sin \theta} \right]$$

$$= 2U \left[\sin(\alpha + \theta - \varepsilon) + \sin(\alpha + \varepsilon_0) \right] \left[\frac{\left(1-r+\frac{d\varepsilon}{d\theta}\right)(1+r)}{2 \sin \theta} \right]$$

$$\frac{1}{1+r-\frac{d\varepsilon}{d\theta}} = 1-r+\frac{d\varepsilon}{d\theta} + r^2 + \left(\frac{d\varepsilon}{d\theta}\right)^2 + \dots$$

$$\left[\left(1-r+\frac{d\varepsilon}{d\theta}\right) \right] (1+r) = 1+r^2 + \frac{d\varepsilon}{d\theta} + r \frac{d\varepsilon}{d\theta} = \left(1+\frac{d\varepsilon}{d\theta}\right)$$

$\sin(\alpha + \varepsilon_0) = (\alpha + \varepsilon_0)$ [The greatest allowable sum is a small angle.]

$$\begin{aligned} \sin[(\alpha - \varepsilon) + \theta] &= \sin(\alpha - \varepsilon) \cos \theta + \cos(\alpha - \varepsilon) \sin \theta \\ &= (\alpha - \varepsilon) \cos \theta + \sin \theta \end{aligned}$$

$$\begin{aligned} |q_{g'}| &= 2U \left\{ \frac{[(\alpha - \varepsilon) \cos \theta + \sin \theta + (\alpha + \varepsilon_0)] \cdot \left[1 + \frac{d\varepsilon}{d\theta}\right]}{2 \sin \theta} \right\} \\ &= U \left[1 + (\alpha - \varepsilon) \cot \theta + (\alpha + \varepsilon_0) \csc \theta \right] \left[1 + \frac{d\varepsilon}{d\theta} \right] \end{aligned}$$

Making the indicated multiplication and discarding

$$\left[(\alpha - \varepsilon) \cdot \frac{d\varepsilon}{d\theta} \right] \text{ and } \left[(\alpha + \varepsilon_0) \cdot \frac{d\varepsilon}{d\theta} \right]$$

$$|q_{g'}| = U \left[1 + (\alpha - \varepsilon) \cot \theta + (\alpha + \varepsilon_0) \csc \theta + \frac{d\varepsilon}{d\theta} \right] \quad (14)$$

where $\theta = 0$

$$|q_{g'}| = U \left[1 + (\alpha - \varepsilon) \cdot \infty + (\alpha + \varepsilon_0) \cdot \infty + \frac{d\varepsilon}{d\theta} \right]$$

This expression shows that the velocity is infinite at the leading edge for every angle of attack except where

$$-(\alpha - \varepsilon) = (\alpha + \varepsilon_0), \text{ or } \alpha = -\frac{\varepsilon_0 - \varepsilon}{2}$$

For which case

$$|q_{s'}|_{\theta=0} = U \left[1 + (\alpha + \epsilon_0) \lim_{\theta \rightarrow 0} (\csc \theta - \cot \theta) + \left(\frac{d\epsilon}{d\theta} \right)_{\theta=0} \right]$$

$$\begin{aligned} \lim_{\theta \rightarrow 0} (\csc \theta - \cot \theta) &= \lim_{\theta \rightarrow 0} \left(\frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \right) = \lim_{\theta \rightarrow 0} \left(\frac{1 - \cos \theta}{\sin \theta} \right) \\ &= \lim_{\theta \rightarrow 0} \left[\frac{\frac{d}{d\theta} (1 - \cos \theta)}{\frac{d}{d\theta} (\sin \theta)} \right] = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\cos \theta} = 0 \end{aligned}$$

So for the conditions

$$\theta = 0$$

$$\alpha = + \frac{(\epsilon - \epsilon_0)}{2}$$

$$|q_{s'}|_{\theta=0} = U \left[1 + \left(\frac{d\epsilon}{d\theta} \right)_{\theta=0} \right]$$

where θ has the value of π ,

$$\epsilon = \epsilon_0$$

$$|q_{s'}| = U \left[1 + (\alpha + \epsilon_0) \lim_{\theta \rightarrow \pi} (\cot \theta + \csc \theta) + \left(\frac{d\epsilon}{d\theta} \right)_{\epsilon=\epsilon_0} \right]$$

$$\begin{aligned} \lim_{\theta \rightarrow \pi} (\cot \theta + \csc \theta) &= \lim_{\theta \rightarrow \pi} \left(\frac{1 + \cos \theta}{\sin \theta} \right) = \lim_{\theta \rightarrow \pi} \left[\frac{\frac{d}{d\theta} (1 + \cos \theta)}{\frac{d}{d\theta} \sin \theta} \right] \\ &= \lim_{\theta \rightarrow \pi} \frac{-\sin \theta}{\cos \theta} = 0 \end{aligned}$$

and

$$|q_{s'}|_{\theta=\pi} = U \left[1 + \left(\frac{d\epsilon}{d\theta} \right)_{\epsilon=\epsilon_0} \right]$$

For this transformation then the velocity is finite at the trailing edge for all angles of attack but is infinite at the leading edge for all angles of attack save one. The infinite velocity at the leading edge is in violent disagreement with actual flows.

THE ANALYSIS BY THEODORSEN OF THE LIFT OF AIRFOILS OF
ARBITRARY SHAPE.

This method of attacking the problem is similar in one respect to Glauert's analysis in that the transformation from the airfoil to the circle is effected in two steps: first, the airfoil is transformed into a near circle; second, the near circle is transformed into a circle. It is superior to Glauert's method in that the actual airfoil surface is transformed instead of the median line and as a consequence the results are applicable to airfoils of any thickness.

Joukowski's Transformation

$$\zeta = z + \frac{a^2}{z}$$

can be put into a different form. This form is obtained in the following manner.

Substituting $re^{i\theta}$ for z ,

$$\begin{aligned}\zeta = \xi + i\eta &= re^{i\theta} + \frac{a^2}{re^{i\theta}} = re^{i\theta} + \frac{a^2}{r} e^{-i\theta} \\ &= \left[r + \frac{a^2}{r} \right] \cos \theta + i \left[r - \frac{a^2}{r} \right] \sin \theta \\ &= a \left[\left(\frac{r}{a} \right) + \left(\frac{a}{r} \right) \right] \cos \theta + ia \left[\left(\frac{r}{a} \right) - \left(\frac{a}{r} \right) \right] \sin \theta \\ &= a \left[\left(\frac{r}{a} \right)' + \left(\frac{r}{a} \right)^{-1} \right] \cos \theta + ia \left[\left(\frac{r}{a} \right)' - \left(\frac{r}{a} \right)^{-1} \right] \sin \theta\end{aligned}$$

By definition,

$$\cosh \lambda = \frac{e^\lambda + e^{-\lambda}}{2} = \frac{(e^\lambda)' + (e^\lambda)^{-1}}{2}$$

If $\left(\frac{r}{a} \right)$ be substituted for e^λ

$$2 \cosh \lambda = \left(\frac{r}{a} \right)' + \left(\frac{r}{a} \right)^{-1}$$

By definition,

$$\sinh \lambda = \frac{(e^\lambda)' - (e^\lambda)^{-1}}{2}$$

If $\left(\frac{r}{a}\right)$ be substituted for e^λ

$$2 \sinh \lambda = \left(\frac{r}{a}\right)' - \left(\frac{r}{a}\right)^{-1}$$

Therefore,

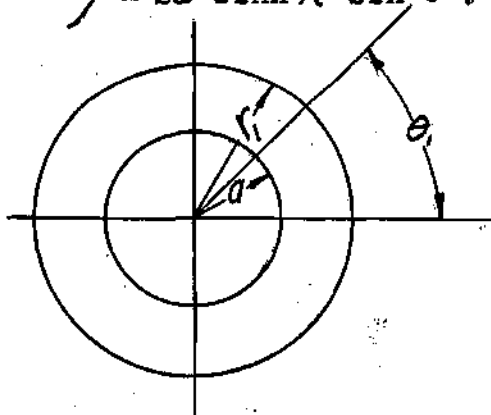
$$z = \xi + i\eta = 2a \cosh \lambda \cos \theta + i 2a \sinh \lambda \sin \theta$$

Equating reals

$$\xi = 2a \cosh \lambda \cos \theta$$

Equating imaginaries,

$$\eta = 2a \sinh \lambda \sin \theta.$$



Let $r = r_1 = \text{constant}$.

Because $e^\lambda = \frac{r}{a}$

$$r_1 = ae^\lambda,$$

$\lambda_1 = \text{constant}.$

$$\xi = 2a \cosh \lambda \cos \theta$$

$$\frac{\xi}{2a \cosh \lambda} = \cos \theta$$

$$\eta = 2a \sinh \lambda \sin \theta$$

$$\frac{\eta}{2a \sinh \lambda} = \sin \theta$$

If the points in the z plane which are to be transformed are confined to the circle of radius r_1 ,

$$\frac{\xi}{2a \cosh \lambda_1} = \cos \theta$$

$$\frac{\eta}{2a \sinh \lambda_1} = \sin \theta$$

Squaring and adding

$$\frac{\xi^2}{4a^2 \cosh^2 \lambda_1} + \frac{\eta^2}{4a^2 \sinh^2 \lambda_1} = \cos^2 \theta + \sin^2 \theta = 1.$$

This equation is in the form of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

It is, therefore, an ellipse in the \mathcal{J} plane. We see that points on a circle in the z plane transform to points on an ellipse in the \mathcal{J} plane. Points on a circle of radius r_2 in the z plane transform to the ellipse in the \mathcal{J} plane, the equation for which is

$$\frac{\xi^2}{4a^2 \cosh^2 \lambda_2} + \frac{\eta^2}{4a^2 \sinh^2 \lambda_2} = 1.$$

If the circle has the radius r_3 , the equation of the ellipse is

$$\frac{\xi^2}{4a^2 \cosh^2 \lambda_3} + \frac{\eta^2}{4a^2 \sinh^2 \lambda_3} = 1.$$

Hence every circle in the z plane with center at the origin transforms into an ellipse in the \mathcal{J} plane. All the circles have the same center which makes them concentric.

The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has foci on the x axis at

$$x' = \pm \sqrt{a^2 - b^2}.$$

The foci of the ellipses, which are the transforms in the \mathcal{J} plane of the concentric circles in the z plane, are located on the ξ axis at

$$\begin{aligned}\xi' &= \pm \sqrt{(2a \cosh \lambda)^2 - (2a \sinh \lambda)^2} \\ &= \pm 2a \sqrt{\cosh^2 \lambda - \sinh^2 \lambda} \\ &= \pm 2a. \quad [\cosh^2 \lambda - \sinh^2 \lambda = 1]\end{aligned}$$

These ellipses are confocal.

Let $\theta = \text{constant} = \theta$,

$$\begin{aligned}\frac{\xi^2}{\cos^2 \theta} &= 4a^2 \cosh^2 \lambda \\ \frac{\eta^2}{\sin^2 \theta} &= 4a^2 \sinh^2 \lambda\end{aligned}$$

Hence

$$\frac{\xi^2}{\cos^2 \theta} - \frac{\eta^2}{\sin^2 \theta} = 4a^2 (\cosh^2 \lambda - \sinh^2 \lambda) = 4a^2$$

and

$$\frac{\xi^2}{(2a \cos \theta)^2} - \frac{\eta^2}{(2a \sin \theta)^2} = 1$$

Hence a straight line through the origin at an angle θ , with the x axis transforms into an hyperbola in the \mathcal{J} plane.

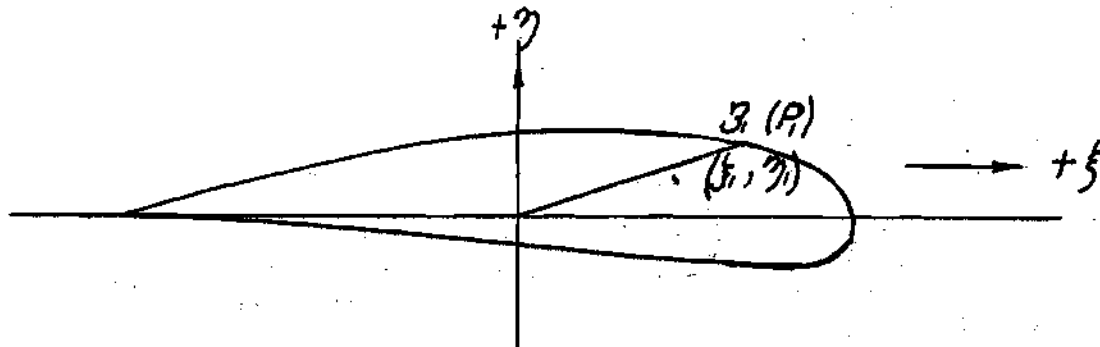
Hence using θ as a parameter

$$\frac{(\xi)^2}{(2a \cos \theta)^2} - \frac{(\eta)^2}{(2a \sin \theta)^2} = 1$$

This is the equation for a family of hyperbolas in the \mathcal{J} plane with common foci at

$$\xi = \sqrt{(2a \cos \theta)^2 + (2a \sin \theta)^2} = \pm 2a.$$

We now have sufficient relations for transforming any point in ζ plane to the corresponding point in z plane.



Transform the point P , to the z plane.

We will first find r , by the following method,

$$\frac{(\zeta_1)^2}{(2a \cosh \lambda)^2} + \frac{\eta_1^2}{(2a \sinh \lambda)^2} = 1$$

$$4a^2 \sinh^2 \lambda \zeta_1^2 + 4a^2 (1 + \sinh^2 \lambda) \eta_1^2 = 4a^2 (1 + \sinh^2 \lambda) 4a^2 \sinh^2 \lambda$$

$$4a^2 \sinh^4 \lambda + (4a^2 - \zeta_1^2 - \eta_1^2) \sinh^2 \lambda - \eta_1^2 = 0$$

$$\sinh^2 \lambda = \frac{-(4a^2 - \zeta_1^2 - \eta_1^2) + \sqrt{(4a^2 - \zeta_1^2 - \eta_1^2)^2 + 16 a^2 \eta_1^2}}{8 a^2}$$

This equation gives λ , which determines the radius of the circle, $r = ae^\lambda$, on which the transform of P , lies but does not locate the point on the circle.

The determination of the value of θ , will locate the point on the circle of radius r .

$$\frac{\zeta_1^2}{(2a \cos \theta)^2} - \frac{\eta_1^2}{(2a \sin \theta)^2} = 1$$

Replacing $\cos^2 \theta$ by $(1 - \sin^2 \theta)$ and solving for $\sin \theta$,

$$\sin^2 \theta = \frac{(4a^2 - \zeta_1^2 - \eta_1^2) + \sqrt{(4a^2 - \zeta_1^2 - \eta_1^2)^2 + 16 a^2 \eta_1^2}}{8 a^2}$$

As an illustration of the method let the co-ordinates of the point P , be

$$f_1 = .824$$

$$j_1 = .372$$

Also let the radius of the circle in the z plane which transforms to the real axis in the \mathcal{Z} plane be unity.

$$a = 1$$

Then

$$(4a^2 - f_1^2 - j_1^2) = (4 \times 1 - .824^2 - .372^2) = 3.182$$

$$(4a^2 - f_1^2 - j_1^2)^2 = 10.15$$

$$\sinh^2 \lambda_1 = \frac{-3.182 + \sqrt{10.15 + 16 \times 1 \times .372^2}}{8 \times 1^2} = .041$$

$$= \frac{-3.182 + 3.51}{8} = .041$$

$$\lambda_1 = 0.202$$

$$r_1 = 1 \times e^{0.202} = 1.224$$

$$\sin^2 \theta_1 = \frac{3.182 + 3.51}{8} = 0.8365$$

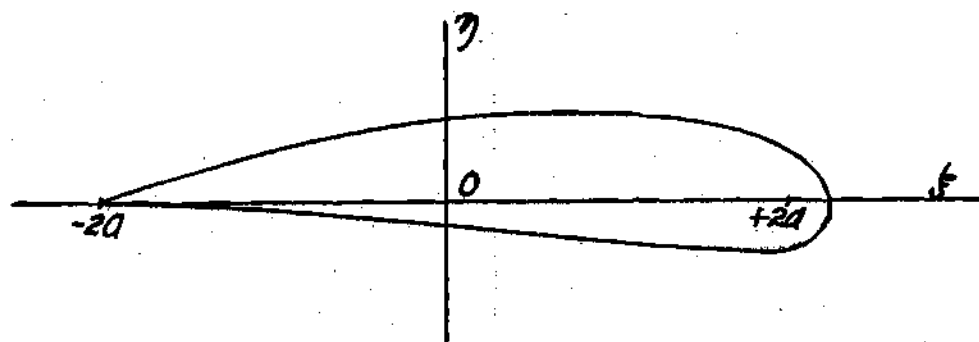
$$\theta_1 = +66^\circ 4'$$

Analysis of signs:

$$\cosh \lambda_1, \cos \theta_1 = \frac{+0.824}{2} = +0.412$$

$$\sinh \lambda_1, \sin \theta_1 = \frac{+0.372}{2} = +0.186$$

$\cosh \lambda$, and $\sinh \lambda$, can not be negative. Therefore, $\sin \theta$ and $\cos \theta$, must both be positive and this condition restricts θ to the first quadrant.



The location of the axis of reals with respect to the airfoil itself is arbitrary. The nearer it is made an axis of symmetry the nearer the curve obtained by the transformation will be like a circle. In order to avoid infinite velocity of the flow at the leading edge the point which is at $\zeta = +2a$ on the axis of reals is placed inside the airfoil. The best location of the leading edge on the axis of reals seems to be

$$\zeta_{LE} = 2a + \frac{R}{2}$$

$$\Im \zeta_{LE} = 0$$

where R = radius of curvature of leading edge.

Because the radius of curvature of the trailing edge is always small or the thickness of the airfoil at the trailing edge is negligible, the trailing edge is taken to be sharp and is put at

$$\zeta = -2a$$

$$\Im \zeta = 0.$$

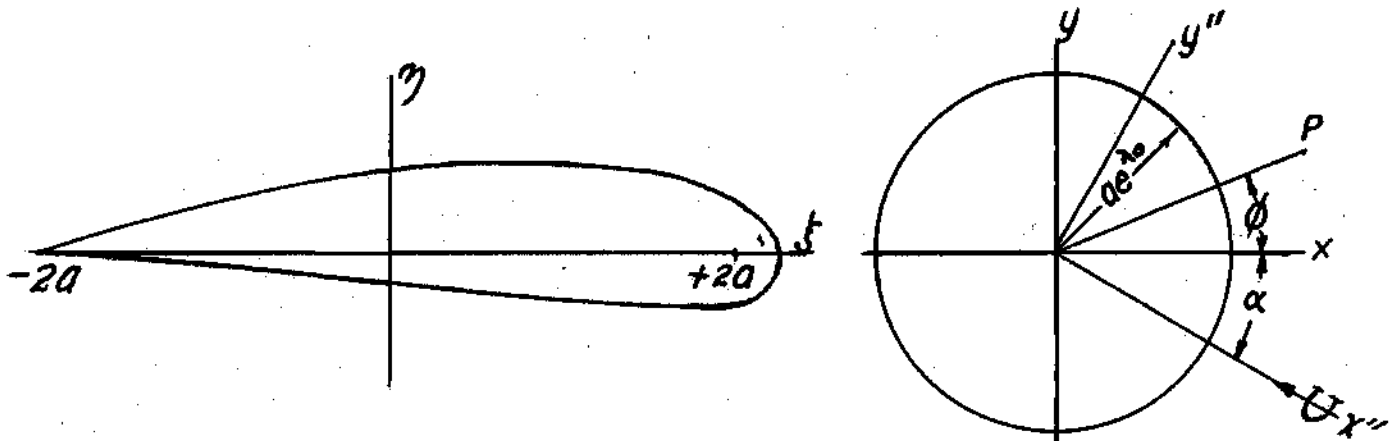
By considering the trailing edge as sharp the circulation can be uniquely determined by Joukowski's hypothesis.

The potential function for the flow about a circle is as has previously been shown

$$w = -U\left(z + \frac{a^2}{z}\right) - i \frac{k}{2\pi} \log \frac{z}{a}$$

The difficulty of the mathematical analysis is lessened if the flow is taken about a circle of radius ae^{λ_0} . This is due to the mathematical set-up brought about by taking

ae^λ for the radius of a circle in the z plane.



The complex co-ordinate of the point P in the z'' plane is,

$$z'' = ae^\lambda e^{i(\phi + \alpha)}$$

Likewise for the z plane,

$$z = ae^\lambda e^{i\phi}$$

Taking the ratio of z'' to z ,

$$\frac{z''}{z} = \frac{ae^\lambda e^{i(\phi + \alpha)}}{ae^\lambda e^{i\phi}} = e^{i\alpha}$$

The potential function for the flow in the z'' plane about the circle of radius ae^{λ_0} is

$$w = -U\left(z'' + \frac{a^2 e^{2\lambda_0}}{z''}\right) - i \frac{k}{2\pi} \log \frac{z''}{ae^{\lambda_0}}$$

Substituting $ze^{i\alpha}$ for z'' , the potential function of the flow in the z plane is

$$w = -U\left(ze^{i\alpha} + \frac{a^2 e^{2\lambda_0}}{ze^{i\alpha}}\right) - i \frac{k}{2\pi} \log \frac{ze^{i\alpha}}{ae^{\lambda_0}}$$

$$\frac{dw}{dz} = -U\left(e^{i\alpha} - \frac{a^2 e^{2\lambda_0}}{z^2 e^{i\alpha}}\right) - \frac{ik}{2\pi} \frac{1}{z}$$

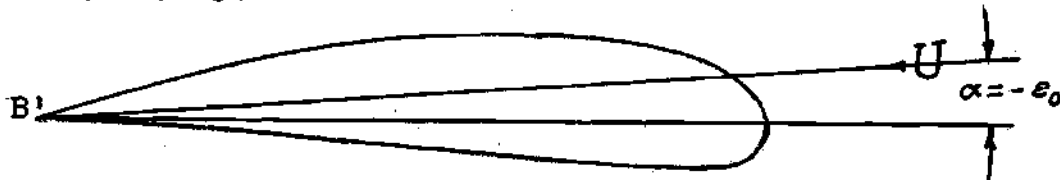
At the surface of the circle

$$\begin{aligned}
 \left(\frac{dw}{dz}\right)_{z=ae^{i\phi}} &= -U \left(\frac{e^{i\alpha}}{e^{i\phi}} - \frac{a^2 e^{2i\alpha}}{a^2 e^{2i\phi}} \right) - \frac{ik}{2\pi} \frac{1}{ae^{i\phi}} \\
 &= -U \left(\frac{e^{i\phi} e^{i\alpha}}{e^{i\phi}} - \frac{e^{-i\phi} e^{-i\alpha}}{e^{i\phi}} \right) - \frac{ik}{2\pi} \frac{1}{ae^{i\phi}} \\
 &= -U \left[\frac{e^{i(\phi+\alpha)}}{e^{i\phi}} - \frac{e^{-i(\phi+\alpha)}}{e^{i\phi}} \right] - \frac{ik}{2\pi} \frac{1}{ae^{i\phi}}
 \end{aligned}$$

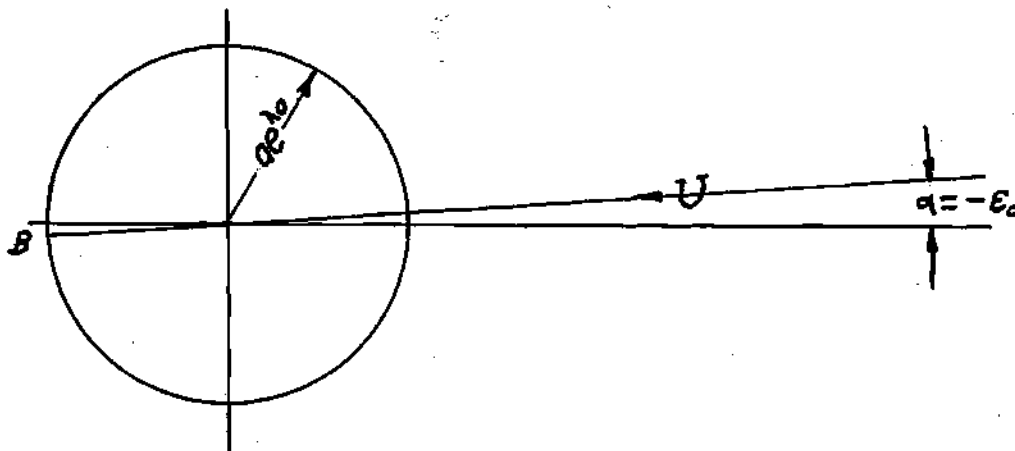
Remembering that

$$\sin(\phi + \alpha) = \frac{e^{i(\phi+\alpha)} - e^{-i(\phi+\alpha)}}{2i}$$

$$\left(\frac{dw}{dz}\right)_{z=ae^{i\phi}} = \frac{-i2U \sin(\phi + \alpha)}{e^{i\phi}} - \frac{ik}{2\pi} \frac{1}{ae^{i\phi}}$$



We have learned from experience that the angle of zero lift for most airfoils is negative.



The flow at infinity for the circle must make the same angle with the axis of reals as it does for the flow in the plane of the airfoil. If the lift of the airfoil is zero, the lift of the circle (really the cylinder) must also be zero. This makes point B a stagnation point of the flow.

Point B will transform to a point on the near circle which will in turn transform to the trailing edge, point B', of the airfoil. In order to avoid infinite velocities at B' on the airfoil, point B on the circle must be a stagnation point for every angle of attack.

The angle of point B is

$$\phi = \pi + \epsilon_0$$

At a stagnation point

$$\frac{dw}{dz} = 0$$

So for point B

$$\left(\frac{dw}{dz}\right)_{z=ae^{\lambda_0}e^{i(\pi+\epsilon_0)}} = 0 = -\frac{12U \sin(\pi + \epsilon_0 + \alpha)}{e^{i\phi}} - \frac{ik}{2\pi} \frac{1}{ae^{\lambda_0}e^{i\phi}}$$

Solving for k,

$$\begin{aligned} k &= -4\pi ae^{\lambda_0} U \sin[\pi + (\alpha + \epsilon_0)] \\ &= 4\pi ae^{\lambda_0} U \sin(\alpha + \epsilon_0) \end{aligned}$$

Substituting this value of k and letting q_c be the velocity at any point on the circle,

$$\begin{aligned} \left(\frac{dw}{dz}\right)_{z=ae^{\lambda_0}e^{i\phi}} &= -\frac{12U \sin(\phi + \alpha)}{e^{i\phi}} - \frac{14\pi ae^{\lambda_0} U \sin(\alpha + \epsilon_0)}{2\pi ae^{\lambda_0} e^{i\phi}} \\ &= -\frac{12U}{e^{i\phi}} [\sin(\phi + \alpha) + \sin(\alpha + \epsilon_0)] \end{aligned}$$

$$\left|\frac{dw}{dz}\right| = |q_c| = 2U [\sin(\phi + \alpha) + \sin(\alpha + \epsilon_0)]$$

We indicate the method of determining the absolute velocity at any point on the surface of the airfoil by

$$|q|_o = |q|_c \left|\frac{dz}{dz'}\right|_o \left|\frac{dz'}{dJ}\right|_o$$

where

$|q|_o$ = Absolute velocity at a point on surface of airfoil.

$|q|_c$ = Absolute velocity at a point on surface of circle.

$\left| \frac{dz}{dz'} \right|_o$ = First derivative of z with respect to z' for the circle of radius ae^{λ_o} .

$\left| \frac{dz'}{d\mathcal{J}} \right|_o$ = First derivative of z' with respect to \mathcal{J} for points on the near circle.

The equation of the circle is

$$z = ae^{\lambda_o} e^{i\phi} = ae^{\lambda_o + i\phi}$$

The equation of the transform of the airfoil, called the near circle, is

$$z' = ae^{\lambda} e^{i\theta} = ae^{\lambda + i\theta}$$

λ will have a different value for each value of θ because the airfoil is not a true ellipse with its foci at $\mathcal{F} = \pm 2a$.

The angle ϕ for a point on the circle will be different from the angle θ of its transform on the near circle.

$$\text{Let } \epsilon = \phi - \theta$$

$$\text{or } \phi = \theta + \epsilon$$

The form of the transformation equation which connects points on the circle with points on the near circle will upon analysis provide a method for computing ϵ , for any value of ϕ , or θ . The transformation equation can be any one which will make this transformation and at the same time preserve the uniform, non-circulating flow at infinity. Its form does not matter at this point of the analysis.

We may write

$$\frac{dz}{dz'} = \frac{dz}{d\theta} \cdot \frac{d\theta}{dz'}$$

Now

$$\frac{dz}{d\theta} = ae^{\lambda_0} e^{i\phi} \cdot i \frac{d\phi}{d\theta} = iz \frac{d\phi}{d\theta}$$

and

$$\frac{d\phi}{d\theta} = 1 + \frac{d\epsilon}{d\theta}$$

Substituting this value of $\frac{d\phi}{d\theta}$

$$\frac{dz}{d\theta} = iz \left(1 + \frac{d\epsilon}{d\theta}\right).$$

Likewise

$$\frac{dz'}{d\theta} = ae^{\lambda + i\theta} \left[\frac{d\lambda}{d\theta} + i \cdot 1 \right] = z' \left(\frac{d\lambda}{d\theta} + i \right)$$

or

$$\frac{d\theta}{dz'} = \frac{1}{z' \left(\frac{d\lambda}{d\theta} + i \right)}$$

which makes

$$\frac{dz}{dz'} = \frac{iz \left(1 + \frac{d\epsilon}{d\theta}\right)}{z' \left(\frac{d\lambda}{d\theta} + i \right)} = \frac{z}{z'} \cdot \frac{\left(1 + \frac{d\epsilon}{d\theta}\right)}{i \left(1 + \frac{d\lambda}{d\theta}\right)} = \frac{z}{z'} \cdot \frac{\left(1 + \frac{d\epsilon}{d\theta}\right)}{\left(1 - i \frac{d\lambda}{d\theta}\right)}$$

or

$$\begin{aligned} \frac{dz}{dz'} &= \frac{z}{z'} \frac{\left(1 + \frac{d\epsilon}{d\theta}\right) \left(1 + i \frac{d\lambda}{d\theta}\right)}{\left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]} \\ \left| \frac{dz}{dz'} \right|^2 &= \left| \frac{z}{z'} \right| \frac{\left(1 + \frac{d\epsilon}{d\theta}\right) \left(1 + i \frac{d\lambda}{d\theta}\right)}{\left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]} \cdot \left| \frac{z}{z'} \right| \frac{\left(1 + \frac{d\epsilon}{d\theta}\right) \left(1 - i \frac{d\lambda}{d\theta}\right)}{\left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]} \\ &= \left| \frac{z}{z'} \right|^2 \frac{\left(1 + \frac{d\epsilon}{d\theta}\right)^2 \left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]}{\left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]^2} = \left| \frac{z}{z'} \right|^2 \frac{\left(1 + \frac{d\epsilon}{d\theta}\right)^2}{\left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]} \end{aligned}$$

Taking the square root

$$\left| \frac{dz}{dz'} \right| = \frac{ae^{\lambda_0} \left[1 + \frac{d\epsilon}{d\theta}\right]}{ae^{\lambda} \left[1 + \left(\frac{d\lambda}{d\theta}\right)^2\right]^{\frac{1}{2}}}$$

Joukowski's transformation for transforming from airfoil to near circle is

$$z = z' + \frac{a^2}{z'}$$

Hence,

$$\frac{dz}{dz'} = 1 - \frac{a^2}{z'^2}$$

For points on near circle

$$\begin{aligned} \left(\frac{dz}{dz'}\right)_0 &= 1 - \frac{a^2}{a^2 e^{2\lambda} e^{i2\theta}} \\ &= \frac{e^\lambda e^{i\theta}}{e^\lambda e^{i\theta}} - \frac{e^{-\lambda} e^{-i\theta}}{e^\lambda e^{i\theta}} \\ &= \frac{1}{e^\lambda e^{i\theta}} \left[e^{\lambda+i\theta} - e^{-(\lambda+i\theta)} \right] \\ &= \frac{2 \sinh(\lambda+i\theta)}{e^{\lambda+i\theta}} \\ &= 2 \left[\frac{\sinh \lambda \cosh i\theta + \cosh \lambda \sinh i\theta}{e^{\lambda+i\theta}} \right] \end{aligned}$$

$$\text{But } \cosh i\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos \theta$$

$$\text{also } \sinh i\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = i \sin \theta$$

Substituting for $\cosh i\theta$ and $\sinh i\theta$

$$\begin{aligned} \left(\frac{dz}{dz'}\right)_0 &= \frac{2 \sinh \lambda \cos \theta + i \cosh \lambda \sin \theta}{e^{\lambda+i\theta}} \\ \left|\frac{dz}{dz'}\right|_0^2 &= 2 \left[\frac{\sinh \lambda \cos \theta + i \cosh \lambda \sin \theta}{e^{\lambda+i\theta}} \right] 2 \left[\frac{\sinh \lambda \cos \theta - i \cosh \lambda \sin \theta}{e^{\lambda-i\theta}} \right] \\ &= \frac{4 [\sinh^2 \lambda \cos^2 \theta + \cosh^2 \lambda \sin^2 \theta]}{e^{2\lambda}} \end{aligned}$$

Substituting

$$\sinh^2 \lambda + 1 = \cosh^2 \lambda$$

$$\begin{aligned} \left|\frac{dz}{dz'}\right|_0^2 &= \frac{4 [\sinh^2 \lambda \cos^2 \theta + \sinh^2 \lambda \sin^2 \theta + \sin^2 \theta]}{e^{2\lambda}} \\ &= 4 \left[\frac{\sinh^2 \lambda (\cos^2 \theta + \sin^2 \theta) + \sin^2 \theta}{e^{2\lambda}} \right] \end{aligned}$$

$$= 4 \left[\frac{\sinh^2 \lambda + \sin^2 \theta}{e^{2\lambda}} \right]$$

Taking the square root

$$\left| \frac{dz}{dz'} \right|_0 = \frac{2 [\sinh^2 \lambda + \sin^2 \theta]^{1/2}}{e^{\lambda}}$$

Substituting for $\left| \frac{dz}{dz'} \right|_0$ and $\left| \frac{dz'}{dz} \right|_0$

$$\begin{aligned} |q|_0 &= kU [\sin(\phi + \alpha) + \sin(\alpha + \epsilon_0)] \cdot \frac{e^{\lambda_0} \left[1 + \frac{d\epsilon}{d\theta} \right]}{e^{\lambda} \left[1 + \left(\frac{d\lambda}{d\theta} \right)^2 \right]^{1/2} 2 [\sinh^2 \lambda + \sin^2 \theta]^{1/2}} \\ &= \frac{e^{\lambda_0} U [\sin(\phi + \alpha) + \sin(\alpha + \epsilon_0)] \left[1 + \frac{d\epsilon}{d\theta} \right]}{\left[1 + \left(\frac{d\lambda}{d\theta} \right)^2 \right]^{1/2} [\sinh^2 \lambda + \sin^2 \theta]^{1/2}} \end{aligned}$$

substituting

$$\phi = \theta + \epsilon$$

and taking the velocity at infinity as V instead of U , the absolute value of the velocity at any point on the surface of the airfoil is

$$|q|_0 = \frac{Ve^{\lambda_0} [\sin(\alpha + \theta + \epsilon) + \sin(\alpha + \epsilon_0)] \left[1 + \frac{d\epsilon}{d\theta} \right]}{\left[1 + \left(\frac{d\lambda}{d\theta} \right)^2 \right]^{1/2} [\sinh^2 \lambda + \sin^2 \theta]^{1/2}}$$

This expression which gives the absolute value of the velocity at any point on the airfoil is one of pure elegance.

because to this point no approximation has been made in order to simplify the analysis. Before it can be used, however, it will be necessary to compute the value of λ_0 and the special values of ϵ , $\frac{d\epsilon}{d\theta}$ and $\frac{d\lambda}{d\theta}$ at the point on the airfoil at which the velocity is desired. By computing q_c for every point on the airfoil for a chosen value of α a pressure distribution diagram for the entire airfoil may be plotted by application

of Bernoulli's theorem for that angle of attack. For experimental verification of this pressure diagram one must determine the pressures on an airfoil which as nearly as possible simulates one of infinite aspect ratio.

To compute the factors λ_0 , etc. which are needed before a single value of $|q_0|$ can be computed it is necessary to know and to analyse the function which is used to transform the points from the near circle to the circle. Theodorsen has used

$$z' = ze^{\sum_{n=1}^{\infty} (A'_n + iB'_n) \frac{1}{z^n}}$$

For very large values of z , $\frac{1}{z} = 0$ and

$$z' = ze^0 = z$$

Therefore, the flow at infinity is preserved.

Previously we have shown that

$$\frac{z'}{z} = \frac{ae^{\lambda_0 e^{i\theta}}}{ae^{\lambda_0 e^{i\phi}}} = e^{\lambda - \lambda_0} \cdot e^{i(\theta - \phi)} = e^{(\lambda - \lambda_0) + i(\theta - \phi)}$$

The transformation is

$$\frac{z'}{z} = e^{\sum_{n=1}^{\infty} (A'_n + iB'_n) \frac{1}{z^n}} = e^{\sum_{n=1}^{\infty} (A'_n + iB'_n) \frac{(\cos n\phi - i \sin n\phi)}{a^n e^{n\lambda_0}}}$$

Therefore, letting $A_n = \frac{A'_n}{e^{n\lambda_0}}$ and $B_n = \frac{B'_n}{e^{n\lambda_0}}$

$$e^{(\lambda - \lambda_0) + i(\theta - \phi)} = e^{\sum_{n=1}^{\infty} \left[\frac{A_n}{a^n} \cos n\phi + \frac{B_n}{a^n} \sin n\phi \right] + i \left[\frac{B_n}{a^n} \cos n\phi - \frac{A_n}{a^n} \sin n\phi \right]}$$

Equating exponents

$$(\lambda - \lambda_0) + i(\theta - \phi) = \sum_{n=1}^{\infty} \left[\frac{A_n}{a^n} \cos n\phi - \frac{B_n}{a^n} \sin n\phi \right] + i \left[\frac{B_n}{a^n} \cos n\phi - \frac{A_n}{a^n} \sin n\phi \right]$$

Equating reals

$$\lambda - \lambda_0 = \sum_{n=1}^{\infty} \left[\frac{A_n}{a^n} \cos n\phi + \frac{B_n}{a^n} \sin n\phi \right]$$

Equating imaginaries

$$\theta - \phi = \sum_1^{\infty} \left[\frac{B_n}{a^n} \cos n\phi - \frac{A_n}{a^n} \sin n\phi \right]$$

Multiplying both sides of the equation for λ by $d\phi$ and indicating integration from 0 to 2π as the limits of ϕ ,

$$\int_0^{2\pi} \lambda d\phi = \lambda_0 \int_0^{2\pi} d\phi + \int_0^{2\pi} \sum_1^{\infty} \left[\frac{A_n}{a^n} \cos n\phi + \frac{B_n}{a^n} \sin n\phi \right] d\phi$$

$$\int_0^{2\pi} \cos n\phi d\phi = \frac{\sin n\phi}{n} \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin n\phi d\phi = -\frac{\cos n\phi}{n} \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sum_1^{\infty} \frac{A_n}{a^n} \cos n\phi d\phi = \int_0^{2\pi} \frac{A_1}{a} \cos \phi d\phi + \int_0^{2\pi} \frac{A_2}{a^2} \cos 2\phi d\phi + \int_0^{2\pi} \frac{A_3}{a^3} \cos 3\phi d\phi + \dots$$

$$\int_0^{2\pi} \sum_1^{\infty} \frac{B_n}{a^n} \sin n\phi d\phi = \int_0^{2\pi} \frac{B_1}{a} \sin \phi d\phi + \int_0^{2\pi} \frac{B_2}{a^2} \sin 2\phi d\phi + \int_0^{2\pi} \frac{B_3}{a^3} \sin 3\phi d\phi + \dots$$

= 0

With these results

$$\int_0^{2\pi} \lambda d\phi = \lambda_0 \int_0^{2\pi} d\phi = 2\pi \lambda_0$$

$$\lambda_0 = \frac{1}{2\pi} \int_0^{2\pi} \lambda d\phi$$

We can say then that λ_0 is the average value of λ around the airfoil.

The values of the different coefficients of the cosine terms may be determined by multiplying both sides of the equation for λ by $\cos k\phi d\phi$ and integrating from 0 to 2π .

$$\int_0^{2\pi} \lambda \cos k\phi d\phi = \lambda_0 \int_0^{2\pi} \cos k\phi d\phi + \int_0^{2\pi} \left[\sum_1^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos k\phi d\phi$$

$$+ \int_0^{2\pi} \left[\sum_1^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos k\phi d\phi$$

The meaning of this is:

For $k = 1$,

$$\int_0^{2\pi} \lambda \cos \phi \, d\phi = \lambda \int_0^{2\pi} \cos \phi \, d\phi + \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos \phi \, d\phi \\ + \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos \phi \, d\phi$$

For $k = 2$,

$$\int_0^{2\pi} \lambda \cos 2\phi \, d\phi = \lambda \int_0^{2\pi} \cos 2\phi \, d\phi + \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos 2\phi \, d\phi \\ + \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos 2\phi \, d\phi$$

Etc.

For $k = 1$,

$$\int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos \phi \, d\phi = \frac{A_1}{a} \int_0^{2\pi} \cos \phi \cos \phi \, d\phi + \frac{A_2}{a^2} \int_0^{2\pi} \cos 2\phi \cos \phi \, d\phi \\ + \frac{A_3}{a^3} \int_0^{2\pi} \cos 3\phi \cos \phi \, d\phi + \frac{A_4}{a^4} \int_0^{2\pi} \cos 4\phi \cos \phi \, d\phi \\ + \dots$$

We can prove that all the terms on the right side except the first is zero.

Proof:

$$\cos (m+n)\phi = \cos m\phi \cos n\phi - \sin m\phi \sin n\phi$$

$$\cos (m-n)\phi = \cos m\phi \cos n\phi + \sin m\phi \sin n\phi$$

Adding,

$$2 \cos m\phi \cos n\phi = \cos (m+n)\phi + \cos (m-n)\phi$$

Multiplying both sides by $d\phi$ and integrating from 0 to 2π ,

$$\int_0^{2\pi} \cos m\phi \cos n\phi \, d\phi = \left[\frac{1}{2} \frac{\sin(m+n)\phi}{(m+n)} \right]_0^{2\pi} + \left[\frac{1}{2} \frac{\sin(m-n)\phi}{(m-n)} \right]_0^{2\pi}$$

For every pair of integers as 3 & 4, 3 & 7, 8 & 5, etc.,

where the first is m and the second n and the first is

not equal to the second, this integral is zero. In mathe-

mathematical language if

$$\int_0^{2\pi} \cos m\phi \cos n\phi d\phi = 0 \quad m \neq n$$

Because if

$$m + n = p$$

$$m - n = g$$

$$\left[\frac{1}{2} \frac{\sin 2p\pi}{p} - 0 \right] = 0$$

$$\left[\frac{1}{2} \frac{\sin 2g\pi}{g} - 0 \right] = 0$$

But if

$$\begin{aligned} \int_0^{2\pi} \cos^2 m\phi d\phi &= \frac{1}{2} \int_0^{2\pi} \cos 2m\phi d\phi + \frac{1}{2} \int_0^{2\pi} \cos(0)\phi d\phi \\ &= \frac{1}{4m} [\sin 2m\phi]_0^{2\pi} + \frac{1}{2} [\phi]_0^{2\pi} = 0 + \pi \end{aligned} \quad m = n$$

In the first term

$$m = n = 1.$$

In all other terms

$$m = 2, 3, 4, 5, 6, \dots \infty$$

$$n = 1$$

Therefore

$$\int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos \phi d\phi = \frac{A_1}{a} \pi + 0 + 0 + \dots$$

For $k = 1$

$$\int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos \phi d\phi = \frac{B_1}{a} \int_0^{2\pi} \sin \phi \cos \phi d\phi + \frac{B_2}{a^2} \int_0^{2\pi} \sin 2\phi \cos \phi d\phi + \dots$$

Every term on the right side of this equation is equal to zero.

Proof:

$$\sin (m+n)\phi = \sin m\phi \cos n\phi + \cos m\phi \sin n\phi$$

$$\sin (m-n)\phi = \sin m\phi \cos n\phi - \cos m\phi \sin n\phi$$

Adding:

$$\sin m\phi \cos n\phi = \frac{\sin (m+n)\phi}{2} + \frac{\sin (m-n)\phi}{2}$$

Multiplying both sides by $d\phi$ and integrating from 0 to 2π :

$$\int_0^{2\pi} \sin m\phi \cos n\phi d\phi = \frac{1}{2} \left[-\frac{\cos (m+n)\phi}{m+n} \right]_0^{2\pi} + \frac{1}{2} \left[-\frac{\cos (m-n)\phi}{m-n} \right]_0^{2\pi}$$

If

$$m \neq n$$

$$\left[\cos (m+n)\phi \right]_0^{2\pi} = 1 - 1 = 0$$

$$\left[\cos (m-n)\phi \right]_0^{2\pi} = 1 - 1 = 0$$

$$\int_0^{2\pi} \sin m\phi \cos n\phi d\phi = 0$$

If $m = n$

$$\sin m\phi \cos m\phi = \frac{\sin 2m\phi}{2} + \frac{\sin 0}{2}$$

$$\int_0^{2\pi} \sin m\phi \cos m\phi d\phi = \frac{1}{2} \int_0^{2\pi} \sin 2m\phi d\phi = \frac{1}{4m} \left[\cos 2m\phi \right]_0^{2\pi} = 0$$

Therefore,

$$\int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos \phi d\phi = 0$$

$$\lambda_0 \int_0^{2\pi} \cos \phi d\phi = \lambda_0 \sin \phi \Big|_0^{2\pi} = 0$$

We can now conclude that,

$$\int_0^{2\pi} \lambda \cos \phi d\phi = \frac{A_1}{a} \pi$$

or

$$A_1 = \frac{a}{\pi} \int_0^{2\pi} \lambda \cos \phi d\phi$$

We see that a graphical integration is required to determine the value of A_1 . To do this λ must first be obtained for every value of ϕ and the product of $\lambda \cos \phi$ plotted as an ordinate against the corresponding value of ϕ as the abscissae for a sufficient number of values of ϕ . The area under this graph multiplied by $\frac{a}{\pi}$ will be the value of A_1 .

For the general case where

$$\begin{aligned} \int_0^{2\pi} \lambda \cos k\phi \, d\phi &= \lambda \int_0^{2\pi} \cos k\phi \, d\phi + \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos k\phi \, d\phi \\ &\quad + \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos k\phi \, d\phi \\ \int_0^{2\pi} \cos k\phi \, d\phi &= \frac{1}{k} [\sin k\phi]_0^{2\pi} = 0 \end{aligned}$$

Expanding the second term;

$$\begin{aligned} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \cos k\phi \, d\phi &= \frac{A_1}{a} \int_0^{2\pi} \cos \phi \cos k\phi \, d\phi + \frac{A_2}{a^2} \int_0^{2\pi} \cos 2\phi \cos k\phi \, d\phi + \dots \\ &\quad + \frac{A_j}{a^j} \int_0^{2\pi} \cos j\phi \cos k\phi \, d\phi \\ &\quad + \frac{A_k}{a^k} \int_0^{2\pi} \cos k\phi \cos k\phi \, d\phi + \dots \end{aligned}$$

The value of every term on the right side is zero save one.

This one is the k th and we obtain,

$$\frac{A_k}{a^k} \int_0^{2\pi} \cos^2 k\phi \, d\phi = \pi.$$

Expanding the third term of the general case:

$$\begin{aligned} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \cos k\phi \, d\phi &= \frac{B_1}{a} \int_0^{2\pi} \sin \phi \cos k\phi \, d\phi + \frac{B_2}{a^2} \int_0^{2\pi} \sin 2\phi \cos k\phi \, d\phi + \dots \\ &\quad + \frac{B_j}{a^j} \int_0^{2\pi} \sin j\phi \cos k\phi \, d\phi + \frac{B_k}{a^k} \int_0^{2\pi} \sin k\phi \cos k\phi \, d\phi + \dots \end{aligned}$$

The value of every term on the right side is zero.

Therefore,

$$\int_0^{2\pi} \lambda \cos k\phi \, d\phi = \frac{A_k}{a^k} \pi$$

And

$$A_k = \frac{a^k}{\pi} \int_0^{2\pi} \lambda \cos k\phi \, d\phi$$

We can by graphical integration determine as many coefficients as are needed for the degree of precision desired. The k th term can be any term and we can interpret the derived expression to mean,

$$A_1 = \frac{a}{\pi} \int_0^{2\pi} \lambda \cos \phi \, d\phi$$

$$A_2 = \frac{a^2}{\pi} \int_0^{2\pi} \lambda \cos 2\phi \, d\phi$$

$$A_3 = \frac{a^3}{\pi} \int_0^{2\pi} \lambda \cos 3\phi \, d\phi$$

etc.

Multiplying both sides of λ by $\sin k\phi \, d\phi$ and indicating integration from 0 to 2π ,

$$\begin{aligned} \int_0^{2\pi} \lambda \sin k\phi \, d\phi &= \lambda_0 \int_0^{2\pi} \sin k\phi \, d\phi + \int_0^{2\pi} \left[\sum_1^{\infty} \frac{A_n}{a^n} \cos n\phi \right] \sin k\phi \, d\phi \\ &\quad + \int_0^{2\pi} \left[\sum_1^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \sin k\phi \, d\phi \end{aligned}$$

The first term on right side is evidently zero.

In the expansion of the second term on the right side every term will be a product of $\sin m\phi$ and $\cos n\phi$ and we have already seen that the integral of such a product from 0 to 2π is zero.

Expanding the third term,

$$\int_0^{2\pi} \left[\sum_1^{\infty} \frac{B_n}{a^n} \sin n\phi \right] \sin k\phi \, d\phi = \frac{B_1}{a^1} \int_0^{2\pi} \sin \phi \sin k\phi \, d\phi + \frac{B_2}{a^2} \int_0^{2\pi} \sin 2\phi \sin k\phi \, d\phi +$$

$$\frac{B_2}{a^2} \int_0^{2\pi} \sin 3\phi \sin k\phi d\phi + \dots$$

$$+ \frac{B_j}{a^j} \int_0^{2\pi} \sin j\phi \sin k\phi d\phi + \frac{B_K}{a^K} \int_0^{2\pi} \sin^2 k\phi d\phi + \dots$$

We can prove that every term on the right side except the k th term is equal to zero.

$$\cos (m+n)\phi = \cos m\phi \cos n\phi - \sin m\phi \sin n\phi$$

$$\cos (m-n)\phi = \cos m\phi \cos n\phi + \sin m\phi \sin n\phi$$

Subtracting:

$$2 \sin m\phi \sin n\phi = \cos (m-n)\phi - \cos (m+n)\phi$$

$$2 \int_0^{2\pi} \sin m\phi \sin n\phi d\phi = \left[\frac{\sin (m-n)\phi}{(m-n)} - \frac{\sin (m+n)\phi}{(m+n)} \right]_0^{2\pi}$$

Remembering that m and n are integers, whole numbers; substitution of the limits gives zero for every case except where m and n are equal. Where $m = n$, this expression is indeterminate but substitution of $m = n = k$ before integration gives

$$2 \sin k\phi \sin k\phi = \cos 0 - \cos 2k\phi = 1 - \cos 2k\phi$$

And

$$\int_0^{2\pi} \sin^2 k\phi d\phi = \pi$$

Therefore

$$\int_0^{2\pi} \lambda \sin k\phi d\phi = \frac{B_K \pi}{a^K}$$

so

$$B_K = \frac{a^K}{\pi} \int_0^{2\pi} \lambda \sin k\phi d\phi$$

or

$$B_1 = \frac{a}{\pi} \int_0^{2\pi} \lambda \sin \phi d\phi$$

$$B_2 = \frac{a^2}{\pi} \int_0^{2\pi} \lambda \sin 2\phi d\phi$$

$$B_3 = \frac{a^3}{\pi} \int_0^{2\pi} \lambda \sin 3\phi \, d\phi$$

etc.

Computation of $\theta - \phi$.

$$\theta - \phi = \sum_1^{\infty} \left[\frac{B_n}{a^n} \cos n\phi - \frac{A_n}{a^n} \sin n\phi \right]$$

Expanding:

$$\begin{aligned} \theta - \phi = & \left[\frac{B_1}{a} \cos \phi - \frac{A_1}{a} \sin \phi \right] + \left[\frac{B_2}{a^2} \cos 2\phi - \frac{A_2}{a^2} \sin 2\phi \right] \\ & + \left[\frac{B_3}{a^3} \cos 3\phi - \frac{A_3}{a^3} \sin 3\phi \right] + \text{-----} \end{aligned}$$

Suppose we wished to compute the value of θ for $\phi = \frac{\pi}{6}$ the above equation would be,

$$\begin{aligned} [\theta]_{\phi=\frac{\pi}{6}} = & \frac{\pi}{6} + \frac{\cos \frac{\pi}{6}}{\pi} \int_0^{2\pi} \lambda \sin \phi \, d\phi - \frac{\sin \frac{\pi}{6}}{\pi} \int_0^{2\pi} \lambda \cos \phi \, d\phi + \\ & \frac{\cos \frac{\pi}{3}}{\pi} \int_0^{2\pi} \lambda \sin 2\phi \, d\phi - \frac{\sin \frac{\pi}{3}}{\pi} \int_0^{2\pi} \lambda \cos 2\phi \, d\phi + \cos \\ & \frac{\cos \frac{\pi}{2}}{\pi} \int_0^{2\pi} \lambda \sin 3\phi \, d\phi - \frac{\sin \frac{\pi}{2}}{\pi} \int_0^{2\pi} \lambda \cos 3\phi \, d\phi + \text{-----}. \end{aligned}$$

or

$$\begin{aligned} [\theta]_{\phi=\frac{\pi}{6}} = & \frac{\pi}{6} + \frac{1}{\pi} \int_0^{2\pi} \lambda \left\{ \left[\sin \phi \cos \frac{\pi}{6} - \cos \phi \sin \frac{\pi}{6} \right] + \right. \\ & \left. \left[\sin 2\phi \cos \frac{\pi}{3} - \cos 2\phi \sin \frac{\pi}{3} \right] + \text{-----} \right\} d\phi \end{aligned}$$

or

$$\begin{aligned} [\theta]_{\phi=\frac{\pi}{6}} = & \frac{\pi}{6} + \frac{1}{\pi} \int_0^{2\pi} \lambda \left[\sin \left(\phi - \frac{\pi}{6} \right) + \sin 2 \left(\phi - \frac{\pi}{6} \right) + \text{-----} \right] d\phi \\ = & \frac{\pi}{6} + \frac{1}{\pi} \int_0^{2\pi} \lambda \left[\sum_1^{\infty} \sin n \left(\phi - \frac{\pi}{6} \right) \right] d\phi \end{aligned}$$

Let ϕ_c be the value of ϕ for which θ is desired and let θ_c be the desired value of θ .

$$[\theta]_c = \phi_c + \frac{1}{\pi} \int_0^{2\pi} \lambda \left[\sum_1^{\infty} \sin n (\phi - \phi_c) \right] d\phi$$

The difficulty of computation of Θ_c can be lessened if we can get the sum of,

$$\sum_{n=1}^{\infty} \sin n(\phi - \phi_c).$$

This can be done by operating on series with

$$2 \sin m\phi \sin n\phi = \cos (m-n)\phi - \cos (m+n)\phi.$$

Expanding:

$$\sum_{n=1}^{\infty} \sin n(\phi - \phi_c) = \sin (\phi - \phi_c) + \sin 2(\phi - \phi_c) + \sin 3(\phi - \phi_c) + \dots$$

Multiplying and dividing every term by $2 \sin \left(\frac{\phi - \phi_c}{2} \right)$ and

taking out $\frac{1}{2 \sin \left(\frac{\phi - \phi_c}{2} \right)}$:

$$\begin{aligned} \sum_{n=1}^{\infty} \sin n(\phi - \phi_c) &= \frac{1}{2 \sin \left(\frac{\phi - \phi_c}{2} \right)} \left[2 \sin (\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) + \right. \\ &\quad \left. \sin 2(\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) + \sin 3(\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) + \dots \right] \end{aligned}$$

$$\begin{aligned} 2 \sin (\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) &= \cos \left[\phi - \phi_c - \frac{\phi - \phi_c}{2} + \frac{\phi - \phi_c}{2} \right] - \cos \left[\phi - \phi_c + \frac{\phi - \phi_c}{2} - \frac{\phi - \phi_c}{2} \right] \\ &= \cos \left(\frac{\phi - \phi_c}{2} \right) - \cos 3 \left(\frac{\phi - \phi_c}{2} \right) \end{aligned}$$

$$2 \sin 2(\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) = + \cos 3 \left(\frac{\phi - \phi_c}{2} \right) - \cos 5 \left(\frac{\phi - \phi_c}{2} \right)$$

$$2 \sin 3(\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) = + \cos 5 \left(\frac{\phi - \phi_c}{2} \right) - \cos 7 \left(\frac{\phi - \phi_c}{2} \right)$$

$$\begin{aligned} &\dots = \dots \\ &\dots = \dots \end{aligned}$$

$$\begin{aligned} 2 \sin m(\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) &= \cos \left[m\phi - m\phi_c - \frac{\phi - \phi_c}{2} + \frac{\phi - \phi_c}{2} \right] - \cos \left[m\phi - m\phi_c + \frac{\phi - \phi_c}{2} - \frac{\phi - \phi_c}{2} \right] \\ &= \cos \left[(2m-1) \left(\frac{\phi - \phi_c}{2} \right) \right] - \cos \left[(2m+1) \left(\frac{\phi - \phi_c}{2} \right) \right] \end{aligned}$$

$$2 \sin n(\phi - \phi_c) \sin \left(\frac{\phi - \phi_c}{2} \right) = \cos \left[(2n-1) \left(\frac{\phi - \phi_c}{2} \right) \right] - \cos \left[(2n+1) \left(\frac{\phi - \phi_c}{2} \right) \right]$$

Before taking the sum of the first n terms let us examine the m^{th} and n^{th} terms. Let m and n be terms in sequence so that

$$n - m = 1$$

$$n = m + 1.$$

$$\begin{aligned} \text{The sum of these two terms} &= \cos \left[(2m-1) \left(\frac{\phi - \phi_c}{2} \right) \right] - \cos \left[(2m+1) \left(\frac{\phi - \phi_c}{2} \right) \right] \\ &+ \cos \left[(2m+3-1) \left(\frac{\phi - \phi_c}{2} \right) \right] - \cos \left[(2m+3+1) \left(\frac{\phi - \phi_c}{2} \right) \right] \\ &= \cos \left[(2m-1) \left(\frac{\phi - \phi_c}{2} \right) \right] - \cos \left[(2m+3) \left(\frac{\phi - \phi_c}{2} \right) \right] \\ &= \cos \left[(2m-1) \left(\frac{\phi - \phi_c}{2} \right) \right] - \cos \left[(2n+1) \left(\frac{\phi - \phi_c}{2} \right) \right] \end{aligned}$$

This shows that the addition of two of the quantities in sequence leaves the first of the two terms of the first quantity and the second of the two terms of the second quantity. The second term of the first quantity being always equal to and opposite in sign to the first term of the second quantity causes the sum of these two to be zero. Then the sum of the first n terms of our series leaves the first term of the first quantity and the last term of the n^{th} quantity.

Therefore adding the first n terms:

$$\begin{aligned} \sum_1^n \sin n(\phi - \phi_c) &= \frac{1}{2 \sin \left(\frac{\phi - \phi_c}{2} \right)} \left\{ \cos \left(\frac{\phi - \phi_c}{2} \right) - \cos \left[(2n+1) \left(\frac{\phi - \phi_c}{2} \right) \right] \right\} \\ &= \frac{1}{2} \cot \left(\frac{\phi - \phi_c}{2} \right) - \frac{1}{2} \frac{\cos \left[(2n+1) \left(\frac{\phi - \phi_c}{2} \right) \right]}{\sin \left(\frac{\phi - \phi_c}{2} \right)} \end{aligned}$$

We can now write as an approximation,

$$\theta_c = \phi_c + \frac{1}{2\pi} \int_0^{2\pi} \lambda \cot \left(\frac{\phi - \phi_c}{2} \right) d\phi - \frac{1}{2} \int_0^{2\pi} \frac{\lambda \cos \left[(2n+1) \left(\frac{\phi - \phi_c}{2} \right) \right]}{\sin \left(\frac{\phi - \phi_c}{2} \right)} d\phi$$

If n is equal to ∞ it can be proved without much difficulty, by use of integration by parts, that

$$\int_0^{2\pi} \frac{\lambda \cos \left[(2n+1) \left(\frac{\phi - \phi_c}{2} \right) \right]}{\sin \left(\frac{\phi - \phi_c}{2} \right)} d\phi = 0$$

This being true we establish the equation:

$$(\theta)_c = \phi_c + \frac{1}{2\pi} \int_0^{2\pi} \lambda \cot \left(\frac{\phi - \phi_c}{2} \right) d\phi.$$

For an example of the application of this theory to a Clark Y airfoil and a more elegant development see Theordorsen's treatment in N.A.C.A. Report #411, 1932.